GALLAI-EDMONDS DECOMPOSITION OF UNICYCLIC GRAPHS FROM NULL SPACE

L. EMILIO ALLEM, DANIEL A. JAUME, GONZALO MOLINA, MAIKON M. TOLEDO, AND VILMAR TREVISAN

Abstract. Given a unicyclic graph G, we obtain the Gallai-Edmonds decomposition of G using linear algebra tools. More precisely, the Gallai-Edmonds decomposition of G is computed from the null space associated with adjacency matrices of its subtrees.

1. Introduction

The aim of spectral graph theory is to obtain structural properties of a graph using the eigenvalues and eigenvectors of the matrices associated with the graph. In particular, relating classical parameters with spectral parameters is quite useful and this is the general goal of this paper. Here, we use the null space of a unicyclic graph G, which is the null space of its adjacency matrix, to compute the Gallai-Edmonds decomposition of G. More precisely, we obtain the Gallai-Edmonds decomposition (see definition below) of unicyclic graphs from the null decomposition of its subtrees.

Null decomposition of a graph (see definition in Section 2), is defined from subsets of vertices of a graph that satisfy certain properties related to the null space. We remark that the null decomposition provides information on structure of a graph, as for example, matching and independence number [2, 11].

In order to explain our results, we need a few definitions here. For an undirected graph $G = (V, E)$, a matching M in G is a set of pairwise non-adjacent edges. A maximum matching is a matching of largest cardinality in G and the matching number of G, denoted by $\nu(G)$, is the size of a set of any maximum matching. $\mathcal{M}(G)$ denotes the set of all maximum matchings of G. A vertex is saturated by M, if it is an endpoint of one of the edges in the matching M. Otherwise the vertex is said non-saturated. The neighborhood of a vertex $v \in V$ in G is defined to be $N(v) = \{u \in V : \{u, v\} \in E\}$. The Gallai-Edmonds decomposition of G [7, 10] is a partition of V into three sets with certain properties related to maximum matchings of G. More precisely, V is partitioned into three sets $EG(G)$, $R(G)$ and $S(G)$ where:

- $EG(G) = \{v \in V : \exists M \in \mathcal{M}(G)$ such that M does not sature $v\}$.
- $R(G) = N(EG(G)) - EG(G)$, where $N(EG(G)) = \displaystyle\bigcup_{v \in EG(G)} N(v)$.
- $S(G) = V - (EG(G) \cup R(G))$.

For example, consider G the unicyclic graph of Figure 1. We use zig zag edges to represent the edges of a matching. Notice that G has four maximum matchings $\{\{a, b\}, \{c, d\}, \{f, e\}, \{h, i\}\}$, $\{\{a, b\}, \{c, d\}, \{g, e\}, \{h, i\}\}$, $\{\{a, b\}, \{c, i\}, \{f, e\}, \{h, d\}\}$ and $\{\{a, b\}, \{c, i\}, \{g, e\}, \{h, d\}\}$. Thus, the Gallai-Edmonds decomposition of G is given by $EG(G) = \{f, g\}$, $R(G) = \{e\}$ and $S(G) = \{a, b, c, d, i, h\}$.

1991 *Mathematics Subject Classification.* 05C50, 15A18.

Key words and phrases. Unicyclic graph, null space, null decomposition, Gallai-Edmonds decomposition.

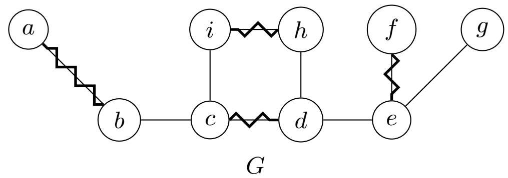

FIGURE 1. Gallai-Edmonds decomposition of the unicyclic graph G.

Gallai-Edmonds decomposition was independently introduced by Gallai [9, 10] and Edmonds [7]. This classical decomposition has several applications [6, 13] and it has been studied by many mathematicians [1, 4, 5, 6, 8, 14, 17] over the years. It is very important in matching theory, given the amount of properties that provides about matchings of any graph, as for example, the well-known result is the Gallai-Edmonds structure theorem ([15], Chapter 3), it states that:

- The subgraph induced by $S(G)$ has a perfect matching.
- $\nu(G) = \frac{1}{2}(|V| - c(EG(G)) + |R(G)|)$, where $c(EG(G))$ denotes the number of components of the graph induced by $EG(G)$.

Cymer [6] shows that the Gallai-Edmonds decomposition can be applied to the pruning methods of constraint programming. The authors of [16, 18] used, respectively, the notions of path-matching and matching cover to obtain more general versions of the Gallai-Edmonds structure theorem for graphs and hypergraphs. For any graph, the Gallai-Edmonds decomposition can be obtained in polynomial time via the Edmonds Matching Algorithm (see [15], Chapters 3 and 9).

One if the goals of this paper is to obtain the Gallai-Edmonds decomposition of a unicyclic graph G using a partition of the vertices based on the null space of its subtrees. Our main contribution is to show how to use linear algebra to obtain such an important decomposition.

The outline of the paper is as follows. In Section 2, we recall some basic definitions and preliminary results. In Sections 3 and 4, we obtain our main results. More precisely, we give a relationship between the Gallai-Edmonds decomposition of a unicyclic graph and the null decomposition of its subtrees. This relationship gives a nice way to compute the Gallai-Edmonds decomposition of unicyclic graphs from the null space of its subtrees, that is, from linear algebra.

2. Basic definitions and preliminary results

The support of a graph G is given by

$$Supp(G) = \{v \in V(G) : \exists x \in \mathcal{N}(G) \text{ such that } x_v \neq 0\}.$$

Where $\mathcal{N}(G)$ denotes the null space of G, that is, the null space of its adjacency matrix. Support gives important information about structure of trees. The next two lemmas show the relationship between the support of trees and their independent sets and matchings.

Lemma 2.1. [11] *Let T be a tree, then $Supp(T)$ is an independent set of T.*

Lemma 2.2. [2] *Let T be a tree, then $EG(T) = Supp(T)$.*

The core of G, denoted by $Core(G)$, is defined:

$$Core(G) = \bigcup_{v \in Supp(G)} N(v).$$

The set of N-vertices of G, denoted by $V(\mathcal{G}_N(G))$, is given by:

$$V(\mathcal{G}_N(G)) = V(G) - (Supp(G) \cup Core(G)).$$

For a tree T, we have that $Core(T) = R(T)$ and $V(\mathcal{G}_N(T)) = S(T)$, see [2, 11]. Null decomposition of G is a pair of induced subgraphs of G. The first subgraph is induced by $Supp(G)$ and $Core(G)$ and the second one is induced by $V(\mathcal{G}_N(G))$. Lemma 2.3 gives a nice way to compute the independence and matching numbers of trees using its null decompositions.

Lemma 2.3. [11] *Let T be a tree. Then*

$$\begin{aligned} \nu(T) &= |Core(T)| + \frac{|V(\mathcal{G}_N(T))|}{2} \\ \alpha(T) &= |Supp(T)| + \frac{|V(\mathcal{G}_N(T))|}{2}. \end{aligned}$$

Let G be a unicyclic graph and let C be the unique cycle of G. For each vertex $v \in V(C)$, we denote by $G\{v\}$ the induced connected subgraph of G with maximum number of vertices, which contains the vertex v and no other vertex of C. $G\{v\}$ is called the pendant tree of G at v (see Figure 3). The unicyclic graph G is said to be of Type I if there exists at least one pendant tree $G\{v\}$ such that $v \notin Supp(G\{v\})$, otherwise, G is said to be of Type II (for more details see [2]). Notice that the set of unicyclic graphs can be divided into two classes, the set of unicyclic graphs of Type I and the set of unicyclic graphs of Type II.

Lemmas 2.4 and 2.5 give a nice way to compute the matching number of unicyclic graphs using the null decomposition of its subtrees.

Lemma 2.4. [2] *If G is a unicyclic graph of Type I and $G\{v\}$ a pendant tree such that $v \notin Supp(G\{v\})$, then*

$$\begin{aligned} \nu(G) &= |Core(G\{v\})| + |Core(G - G\{v\})| + \frac{|V(\mathcal{G}_N(G\{v\}))| + |V(\mathcal{G}_N(G - G\{v\}))|}{2} \\ &= \nu(G\{v\}) + \nu(G - G\{v\}). \end{aligned}$$

Lemma 2.5. [2] *Let G be a unicyclic graph and C its cycle. Let $G - C = \bigcup_{i=1}^{k} T_i$, where T_i is a connected component of $G - C$. If G is a unicyclic graph of Type II, then*

$$\begin{aligned} \nu(G) &= \left\lfloor \frac{|V(C)|}{2} \right\rfloor + \sum_{i=1}^{k} |Core(T_i)| + \frac{|V(\mathcal{G}_N(T_i))|}{2} \\ &= \nu(C) + \sum_{i=1}^{k} \nu(T_i). \end{aligned}$$

In the next sections we will study the Gallai-Edmonds decomposition of unicyclic graphs of Type I and II, respectively.

3. Gallai-Edmonds Decomposition of Unicyclic Graphs of Type I

In this section, we obtain a relationship between the Gallai-Edmonds decomposition of a unicyclic graph G of Type I and the null decompositions of $G\{v\}$ and $G - G\{v\}$, where $v \notin Supp(G\{v\})$.

The next technical lemmas and remarks will be used to prove our main result in this section (Theorem 3.9).

Remark 3.1. *Let G be a unicyclic graph and $G\{v\}$ a pendant tree. Let $u, w \in N(v) \cap V(G - G\{v\})$. Notice that $E(G\{v\})$, $E(G - G\{v\})$ and $\{\{u,v\},\{w,v\}\}$ form a partition of $E(G)$, thus, $E(G) = E(G\{v\}) \cup E(G - G\{v\}) \cup \{\{u,v\},\{w,v\}\}$. Hence, given M a matching in G we have that*

$$M = (M \cap E(G\{v\})) \cup (M \cap E(G - G\{v\})) \cup (M \cap \{\{u,v\},\{w,v\}\}).$$

Remark 3.2. *Note that for all graph G and $v \in V(G)$ we have that*

$$\nu(G) - 1 \leq \nu(G - v) \leq \nu(G).$$

Lemma 3.3. *Let T be a tree and $v \in V(T)$. $v \in Supp(T)$ if, and only if, $\nu(T - v) = \nu(T)$.*

Proof. Suppose that $v \in Supp(T)$, then by Lemma 2.2 there is $M \in \mathcal{M}(T)$ such that M does not saturate v. Thus, M is a matching in $T - v$. Then $\nu(T) = |M| \leq \nu(T - v)$. Given $M \in \mathcal{M}(T - v)$. Note that M is a matching in T, because $T - v \subseteq T$. Thus, $\nu(T - v) = |M| \leq \nu(T)$. Therefore, $\nu(T) = \nu(T - v)$.

Conversely, suppose that $\nu(T) = \nu(T - v)$. Given $M \in \mathcal{M}(T - v)$. Notice that M is a matching in T, because $T - v \subseteq T$. Since $\nu(T) = \nu(T - v)$, we have that $M \in \mathcal{M}(T)$. Moreover, M does not saturate v. Hence, by Lemma 2.2 we conclude that $v \in Supp(T)$. $\square$

Lemma 3.4. *Let G be a unicyclic graph Type I and $G\{v\}$ a pendant tree such that $v \notin Supp(G\{v\})$. If $M \in \mathcal{M}(G)$, then $M \cap E(G - G\{v\}) \in \mathcal{M}(G - G\{v\})$.*

Proof. Suppose there is $M \in \mathcal{M}(G)$ such that $M \cap E(G - G\{v\}) \notin \mathcal{M}(G - G\{v\})$. Thus, we have that

$$|M \cap E(G - G\{v\})| \leq \nu(G - G\{v\}) - 1.$$

Let $u, w \in N(v) \cap V(G - G\{v\})$.
Case 1: $\{u,v\} \notin M$ and $\{w,v\} \notin M$
Since $|M \cap E(G - G\{v\})| \leq \nu(G - G\{v\}) - 1$, we have that

$$\begin{array}{rcl}\nu(G) & = & |M| \\ & = & |M \cap E(G\{v\})| + |M \cap E(G - G\{v\})| \\ & \leq & \nu(G\{v\}) + \nu(G - G\{v\}) - 1 \\ & < & \nu(G\{v\}) + \nu(G - G\{v\}).\end{array}$$

Which is a contradiction, because by Lemma 2.4 we have that $\nu(G) = \nu(G\{v\}) + \nu(G - G\{v\})$.
Case 2: $\{u,v\} \in M$ or $\{w,v\} \in M$
Note that in this case $M \cap E(G\{v\})$ does not saturate v, because v is saturate by $\{u,v\}$ or by $\{w,v\}$. Since $v \notin Supp(G\{v\})$, then by Lemma 2.2 we conclude that $M \cap E(G\{v\}) \notin \mathcal{M}(G\{v\})$. Thus, $|M \cap E(G\{v\})| \leq \nu(G\{v\}) - 1$.

Hence,

$$\begin{aligned}
\nu(G) &= |M| \\
&= |M \cap E(G\{v\})| + |M \cap E(G - G\{v\})| + |M \cap \{\{u,v\},\{w,v\}\}| \\
&= |M \cap E(G\{v\})| + |M \cap E(G - G\{v\})| + 1 \\
&\leq \nu(G\{v\}) - 1 + \nu(G - G\{v\}) - 1 + 1 \\
&\leq \nu(G\{v\}) + \nu(G - G\{v\}) - 1 \\
&< \nu(G\{v\}) + \nu(G - G\{v\}).
\end{aligned}$$

Which is a contradiction, because by Lemma 2.4 we have that $\nu(G) = \nu(G\{v\}) + \nu(G - G\{v\})$. □

Lemma 3.5. *Let G be a unicyclic graph Type I and $G\{v\}$ a pendant tree such that $v \notin Supp(G\{v\})$. Let $u, w \in N(v) \cap V(G - G\{v\})$ and $M \in \mathcal{M}(G)$. If $\{u,v\} \notin M$ and $\{w,v\} \notin M$, then $M \cap E(G\{v\}) \in \mathcal{M}(G\{v\})$, otherwise, $M \cap E(G\{v\}) \in \mathcal{M}(G\{v\} - v)$.*

Proof. **Case 1**: $\{u,v\} \notin M$ and $\{w,v\} \notin M$
Using Remark 3.1 and Lemma 2.4, we have that

$$\nu(G\{v\}) + \nu(G - G\{v\}) = \nu(G) = |M| = |M \cap E(G\{v\})| + |M \cap E(G - G\{v\})|.$$

Since, by Lemma 3.4,

$$|M \cap E(G - G\{v\})| = \nu(G - G\{v\}),$$

we have that

$$\nu(G\{v\}) = |M \cap E(G\{v\})|.$$

Hence, $M \cap E(G\{v\}) \in \mathcal{M}(G\{v\})$.

Case 2: $\{w,v\} \in M$ or $\{u,v\} \in M$
Note that $\{w,v\}$ and $\{u,v\}$ can not belong simultaneously to M, because both are incident in v. Moreover, $M \cap E(G\{v\})$ does not sature v, because $\{w,v\}$ or $\{u,v\}$ is incident in v. Thus, $M \cap E(G\{v\})$ is a matching in $G\{v\} - v$. Using Remark 3.1 and Lemma 2.4 we have that

$$\nu(G\{v\}) + \nu(G - G\{v\}) = \nu(G) = |M| = |M \cap E(G\{v\})| + |M \cap E(G - G\{v\})| + 1.$$

Since, by Lemma 3.4,

$$|M \cap E(G - G\{v\})| = \nu(G - G\{v\}),$$

we have that

$$|M \cap E(G\{v\})| = \nu(G\{v\}) - 1.$$

Therefore, using Lemma 3.3 and Remark 3.2 we conclude that $M \cap E(G\{v\}) \in \mathcal{M}(G\{v\} - v)$. □

Lemma 3.6. *Let G be a unicyclic graph Type I, $G\{v\}$ a pendant tree such that $v \notin Supp(G\{v\})$, $u \in N(v) \cap V(G - G\{v\})$ and $M \in \mathcal{M}(G)$. If $u \notin Supp(G - G\{v\})$, then $\{u,v\} \notin M$.*

Proof. Suppose there is $M \in \mathcal{M}(G)$ such that $\{u,v\} \in M$. Note that $M \cap E(G - G\{v\})$ is matching in $G - G\{v\}$. Moreover, $M \cap E(G - G\{v\})$ does not saturate u. Since $u \notin Supp(G - G\{v\})$, then by Lemma 2.2 we conclude that $M \cap E(G - G\{v\}) \notin \mathcal{M}(G - G\{v\})$. Thus, using Lemma 3.4 we conclude that $M \notin \mathcal{M}(G)$, which is a contradiction. □

Proposition 3.7. *Let G be a unicyclic graph Type I and $G\{v\}$ a pendant tree such that $v \notin Supp(G\{v\})$. Let $u, w \in N(v) \cap V(G - G\{v\})$ such that $u, w \notin Supp(G - G\{v\})$. $M \in \mathcal{M}(G)$ if, and only if, $M = M_1 \cup M_2$, where $M_1 \in \mathcal{M}(G\{v\})$ and $M_2 \in \mathcal{M}(G - G\{v\})$.*

Proof. Given $M \in \mathcal{M}(G)$. Using Lemma 3.6 we conclude that $\{u, v\} \notin M$ and $\{w, v\} \notin M$. Moreover, using Remark 3.1 we have that $M = (M \cap E(G\{v\})) \cup (M \cap E(G - G\{v\}))$. By lemmas 3.5 and 3.4 we have that $M \cap E(G\{v\}) \in \mathcal{M}(G\{v\})$ and $M \cap E(G - G\{v\}) \in \mathcal{M}(G - G\{v\})$.

Conversely, given $M = M_1 \cup M_2$, where $M_1 \in \mathcal{M}(G\{v\})$ and $M_2 \in \mathcal{M}(G - G\{v\})$. Thus, we have that $|M| = |M_1| + |M_2| = \nu(G\{v\}) + \nu(G - G\{v\}) = \nu(G)$. Hence, $M \in \mathcal{M}(G)$. $\square$

Lemma 3.8. [3] *Let G be a unicyclic graph of Type I and $G\{v\}$ a pendant tree such that $v \notin Supp(G\{v\})$. Then $Supp(G\{v\}) \subseteq Supp(G\{v\} - v)$.*

Now, we are able to present the relationship between the Gallai-Edmonds decomposition of a unicyclic graph of Type I and the null decomposition of its subtrees. Theorem 3.9 gives a way to obtain the Gallai-Edmonds decomposition of a unicyclic graph from the null decompositions of $G\{v\}$, $G - G\{v\}$ and $G\{v\} - v$, where $v \notin Supp(G\{v\})$.

Theorem 3.9. *Let G be a unicyclic graph Type I, $G\{v\}$ a pendant tree such that $v \notin Supp(G\{v\})$ and $u, w \in N(v) \cap V(G - G\{v\})$. If $u, w \notin Supp(G - G\{v\})$, then*

(i) $EG(G) = Supp(G\{v\}) \cup Supp(G - G\{v\})$
(ii) $R(G) = Core(G\{v\}) \cup Core(G - G\{v\})$
(iii) $S(G) = V(\mathcal{G}_N(G\{v\})) \cup V(\mathcal{G}_N(G - G\{v\}))$

otherwise,

(i) $EG(G) = Supp(G\{v\} - v) \cup Supp(G - G\{v\})$
(ii) $R(G) = \{v\} \cup Core(G\{v\} - v) \cup Core(G - G\{v\})$
(iii) $S(G) = V(\mathcal{G}_N(G\{v\} - v)) \cup V(\mathcal{G}_N(G - G\{v\}))$

Proof. **Case 1**: $u, w \notin Supp(G - G\{v\})$

(i) Given $x \in EG(G)$. Thus, there is $M \in \mathcal{M}(G)$ such that M does not saturate x. By Proposition 3.7 we conclude that $M = M_1 \cup M_2$, where $M_1 \in \mathcal{M}(G\{v\})$ and $M_2 \in \mathcal{M}(G - G\{v\})$. Note that M_1 and M_2 do not saturate x. Therefore, if $x \in V(G\{v\})$, then by Lemma 2.2 we conclude that $x \in Supp(G\{v\})$. Similarly, if $x \in V(G - G\{v\})$, then we conclude that $x \in Supp(G - G\{v\})$.

Now, given $x \in Supp(G\{v\}) \cup Supp(G - G\{v\})$. We will to obtain $M \in \mathcal{M}(G)$ such that M does not saturate x. If $x \in Supp(G\{v\})$, then by Lemma 2.2 there is $M_1 \in \mathcal{M}(G\{v\})$ such that M_1 does not saturate x. Consider $M_2 \in \mathcal{M}(G - G\{v\})$. Define $M = M_1 \cup M_2$. We have that $|M| = |M_1| + |M_2| = \nu(G\{v\}) + \nu(G - G\{v\}) = \nu(G)$. Hence, $M \in \mathcal{M}(G)$ and M does not saturate x, that is, $x \in EG(G)$. Similarly, if $x \in Supp(G - G\{v\})$, then we define $M = M_1 \cup M_2$, where $M_2 \in \mathcal{M}(G - G\{v\})$ such that M_2 does not saturate x and $M_1 \in \mathcal{M}(G\{v\})$. We conclude that $M \in \mathcal{M}(G)$ and M does not saturate x, thus, $x \in EG(G)$.

(ii) Notice that

$$V(G - G\{v\}) \cap N(V(G\{v\})) = \{u, w\} \text{ and } V(G\{v\}) \cap N(V(G - G\{v\})) = \{v\}.$$

Since $u, w \notin Supp(G - G\{v\})$, then $N(Supp(G\{v\})) \cap V(G - G\{v\}) = \emptyset$ and $N(Supp(G - G\{v\})) \cap V(G\{v\}) = \emptyset$. Moreover, note that

$$\begin{aligned} N(EG(G)) &= N(Supp(G\{v\}) \cup Supp(G - G\{v\})) \\ &= Core(G\{v\}) \cup Core(G - G\{v\}). \end{aligned}$$

By Lemma 2.1 we have that $Supp(G\{v\}) \cup Supp(G - G\{v\})$ is an independent set, thus,

$$\begin{aligned} (Supp(G\{v\}) \cup Supp(G - G\{v\})) \cap (Core(G\{v\}) \cup Core(G - G\{v\})) &= \emptyset \\ EG(G) \cap N(EG(G)) &= \emptyset. \end{aligned}$$

Therefore, $N(EG(G)) - EG(G) = R(G) = Core(G\{v\}) \cup Core(G - G\{v\})$.

(iii) Just use the items (i) and (ii).

Case 2: $u \in Supp(G - G\{v\})$ or $w \in Supp(G - G\{v\})$

(i) Given $x \in EG(G)$. Thus, there is $M \in \mathcal{M}(G)$ such that M does not saturate x. Note that in this case we can have $\{u, v\} \in M$ or $\{w, v\} \in M$. If $\{u, v\} \in M$ or $\{w, v\} \in M$, then by Lemma 3.5 $M \cap E(G\{v\}) \in \mathcal{M}(G\{v\} - v)$. Moreover, by Lemma 3.4 we have that $M \cap E(G - G\{v\}) \in \mathcal{M}(G - G\{v\})$. Notice that $M \cap E(G\{v\})$ and $M \cap E(G - G\{v\})$ do not saturate x. Thus, if $x \in V(G\{v\} - v)$, then using Lemma 2.2 we conclude that $x \in Supp(G\{v\} - v)$. Similarly, if $x \in V(G - G\{v\})$, then we conclude that $x \in Supp(G - G\{v\})$.

If $\{u, v\} \notin M$ and $\{w, v\} \notin M$, then by same argument as in item (i) of the Case 1 we have that $x \in Supp(G\{v\}) \cup Supp(G - G\{v\})$. Thus, using Lemma 3.8 we have that $x \in Supp(G\{v\} - v) \cup Supp(G - G\{v\})$.

Now, given $x \in Supp(G\{v\} - v) \cup Supp(G - G\{v\})$. We will to obtain $M \in \mathcal{M}(G)$ such that M does not saturate x. If $x \in Supp(G\{v\} - v)$ and $u \in Supp(G - G\{v\})$, then by Lemma 2.2 there are $M_1 \in \mathcal{M}(G\{v\} - v)$ and $M_2 \in \mathcal{M}(G - G\{v\})$ such that M_1 and M_2 do not saturate x and u, respectively. We define $M = M_1 \cup M_2 \cup \{u, v\}$. We have that $|M| = |M_1| + |M_2| + 1 = \nu(G\{v\} - v) + \nu(G - G\{v\}) + 1 = \nu(G\{v\}) + \nu(G - G\{v\}) = \nu(G)$. Therefore, $M \in \mathcal{M}(G)$ and M does not saturate x. Similarly, if $x \in Supp(G\{v\} - v)$ and $w \in Supp(G - G\{v\})$, then we define $M = M_1 \cup M_2 \cup \{w, v\}$, where $M_1 \in \mathcal{M}(G\{v\} - v)$ and $M_2 \in \mathcal{M}(G - G\{v\})$ and M_1 and M_2 do not saturate x and w, respectively. Hence, $M \in \mathcal{M}(G)$ and M does not saturate x. If $x \in Supp(G - G\{v\})$, then we define $M = M_1 \cup M_2$, where $M_1 \in \mathcal{M}(G\{v\})$ and $M_2 \in \mathcal{M}(G - G\{v\})$ and M_1 does not saturate x. Therefore, $M \in \mathcal{M}(G)$ and M does not saturate x.

(ii) Note that

$$\begin{aligned} N(EG(G)) &= N(Supp(G\{v\} - v) \cup Supp(G - G\{v\})) \\ &= \{v\} \cup Core(G\{v\} - v) \cup Core(G - G\{v\}). \end{aligned}$$

Moreover $N(EG(G)) \cap EG(G) = \emptyset$. Therefore, $N(EG(G)) - EG(G) = R(G) = \{v\} \cup Core(G\{v\} - v) \cup Core(G - G\{v\})$.

(iii) Just use the items (i) and (ii).

□

In the following example, we use the Theorem 3.9 to obtain the Gallai-Edmonds decomposition of the unicyclic graph G in Figure 2. Analyzing the entries of the vectors of the basis of the $\mathcal{N}(G\{v\})$ and $\mathcal{N}(G - G\{v\})$ we obtain that $Supp(G\{v\}) = \{z, x, k\}$,

$Core(G\{v\}) = \{v\}$, $V(\mathcal{G}_N(G\{v\})) = \{c, y\}$, $Supp(G-G\{v\}) = \{e, d, i, g, h, m, n, o, p, s, t\}$, $Core(G - G\{v\}) = \{a, j, l, q, r\}$ and $V(\mathcal{G}_N(G - G\{v\})) = \{u, f, w, b\}$.

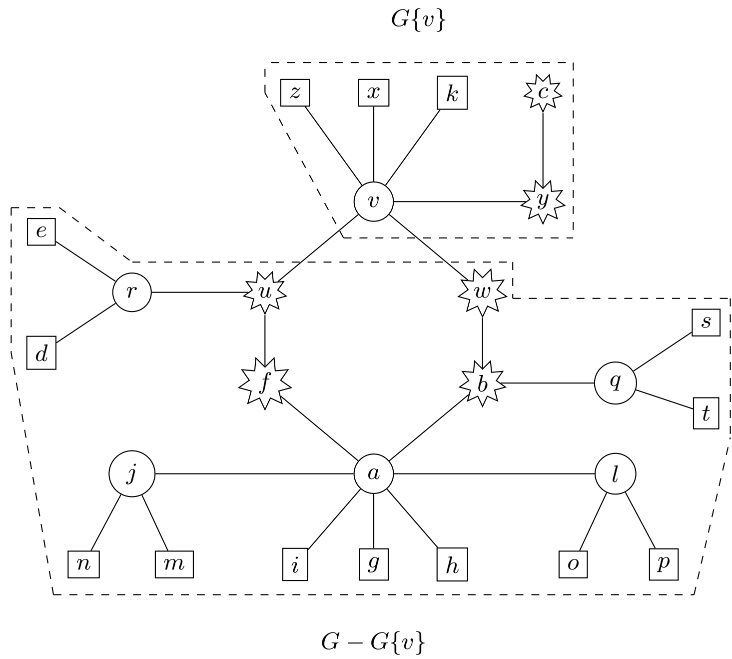

FIGURE 2. Unicyclic graph of Type I and its subtrees $G\{v\}$ and $G - G\{v\}$.

Note that G is a unicyclic graph of Type I, because $v \notin Supp(G\{v\})$. Moreover, $u, w \notin Supp(G - G\{v\})$. Therefore, by Theorem 3.9 the Gallai-Edmonds decomposition of G is given by:

$$\begin{aligned} EG(G) &= Supp(G\{v\}) \cup Supp(G - G\{v\}) = \{z, x, k, e, d, i, g, h, m, n, o, p, s, t\}, \\ R(G) &= Core(G\{v\}) \cup Core(G - G\{v\}) = \{a, j, l, q, r, v\}, \\ S(G) &= V(\mathcal{G}_N(G\{v\})) \cup V(\mathcal{G}_N(G - G\{v\})) = \{u, f, w, b, c, y\}. \end{aligned}$$

4. Gallai-Edmonds decomposition of Unicyclic Graphs of Type II

In this section, we obtain a relationship between the Gallai-Edmonds decomposition of a unicyclic graph G of Type II and the null decompositions of $G - C$ or $G\{v\}$, where C is the unique cycle of G and $v \in V(C)$.

Next, we will define the set of intermediate edges. We study under what circumstance these edges are in a maximum matching of G (see Lemma 4.4).

Definition 4.1. *Let G be a unicyclic graph and C its cycle. The set of intermediate edges, denoted by $\mathcal{IE}(G)$, is defined as $\mathcal{IE}(G) = E(G) - (E(C) \cup E(G - C))$.*

The following lemmas and remarks will be crucial to prove our main results in this section (Theorems 4.6 and 4.8).

Remark 4.2. *Let G be a unicyclic graph and C its cycle. Notice that $E(C)$, $E(G - C)$ and $\mathcal{IE}(G)$ form a partition of $E(G)$, thus, $E(G) = E(C) \cup E(G - C) \cup \mathcal{IE}(G)$.*

Therefore, given a matching M in G we have that

$$\begin{aligned} M &= (M \cap E(C)) \cup (M \cap E(G-C)) \cup (M \cap \mathcal{IE}(G)) \\ &= (M \cap E(C)) \cup \bigcup_{v \in V(C)} M \cap E(G\{v\}). \end{aligned}$$

Lemma 4.3. *Let G be a unicyclic graph of Type II and C its cycle. If $M \in \mathcal{M}(G)$, then $M \cap E(C) \in \mathcal{M}(C)$ and $M \cap E(G\{v\}) \in \mathcal{M}(G\{v\})$ for all $v \in V(C)$.*

Proof. Suppose there is $M \in \mathcal{M}(G)$ such that $M \cap E(C) \notin \mathcal{M}(C)$ or $M \cap E(G\{w\}) \notin \mathcal{M}(G\{w\})$ for some $w \in V(C)$. That is, $|M \cap E(C)| \leq \nu(C) - 1$ or $|M \cap E(G\{w\})| \leq \nu(G\{w\}) - 1$. Note that given $u \in V(C)$ we have that $u \in Supp(G\{u\})$, then by Lemma 3.3 $\nu(G\{u\}) = \nu(G\{u\} - u)$. Consider $G - C = \bigcup_{i=1}^{k} T_i$, where T_i is a connected component of $G - C$. Thus, we have that

$$\begin{aligned} \nu(G) &= |M| \\ &= |M \cap E(C)| + \sum_{u \notin V(C)} |M \cap E(G\{u\})| \text{ (Remark 4.2)} \\ &= |M \cap E(C)| + |M \cap E(G\{w\})| + \sum_{u \in V(C)-\{w\}} |M \cap E(G\{u\})| \\ &\leq +\nu(C) + \nu(G\{w\}) - 1 + \sum_{u \in V(C)-\{w\}} \nu(G\{u\}) \\ &= -1 + \nu(C) + \sum_{u \in V(C)} \nu(G\{u\}) \\ &= -1 + \nu(C) + \sum_{u \in V(C)} \nu(G\{u\} - u) \\ &= -1 + \nu(C) + \sum_{i=1}^{k} \nu(T_i) \\ &< \nu(C) + \sum_{i=1}^{k} \nu(T_i). \end{aligned}$$

Which is a contradiction, because by Lemma 2.5 $\nu(G) = \nu(C) + \sum_{i=1}^{k} \nu(T_i)$. □

Lemma 4.4. *Let G be a unicyclic graph Type II, C its cycle and $M \in \mathcal{M}(G)$. Let $e \in \mathcal{IE}(G)$. If $|V(C)|$ is even, then $e \notin M$.*

Proof. Let $e \in \mathcal{IE}(G)$. Suppose there is $M \in \mathcal{M}(G)$ such that $e \in M$. Consider $G - C = \bigcup_{i=1}^{k} T_i$, where T_i is a connected component of $G - C$. Define $e = \{u, v\}$ such that $u \in V(T_j)$ for some $j \in \{1, 2, \ldots, k\}$ and $v \in V(C)$. Note that $M \cap E(C)$ is a matching in C. Moreover, $M \cap E(C)$ do not saturate v, because v is saturate by e. As $|V(C)|$ is even and $\nu(C) = \frac{|V(C)|}{2}$, then all maximum matchings in C are perfect matching. Therefore, $M \cap E(C) \notin \mathcal{M}(C)$. Hence, using Lemma 4.3 we have that $M \notin \mathcal{M}(G)$, which is a contradiction. □

Lemma 4.5. [2] *Let G be a unicyclic graph and C its cycle. Let $G\{v\}$ be a pendant tree such that $v \in Supp(G\{v\})$. If $u \in N(v) \cap V(G\{v\})$, then $u \notin Supp(G-C)$.*

We are now ready to prove our main results in this section. Theorems 4.6 and 4.8 provide a way to obtain the Gallai-Edmonds decomposition of a unicyclic graph of Type II from the null decompositions of $G-C$ and $G\{v\}$, respectively.

Theorem 4.6. *Let G be a unicyclic graph of Type II and C its cycle. If $|V(C)|$ is even, then*

(i) $EG(G) = Supp(G-C)$
(ii) $R(G) = Core(G-C)$
(iii) $S(G) = V(\mathcal{G}_N(G-C)) \cup V(C)$

Proof. (i) Consider $G-C = \bigcup\limits_{i=1}^{k} T_i$, where T_i is a connected component of $G-C$. Given $x \in EG(G)$. Thus, there is $M \in \mathcal{M}(G)$ such that M does not saturate x. Since $|V(C)|$ is even, then using Remark 4.2 and Lemma 4.4 we conclude that

$$M = (M \cap E(C)) \cup \left(\bigcup_{i=1}^{k} (M \cap E(T_i)) \right).$$

Note that by Lemma 4.3 we have that $M \cap E(C) \in \mathcal{M}(C)$. Moreover, we have that $M \cap E(T_i) \in \mathcal{M}(T_i)$, otherwise

$$\nu(G) = |M| < \nu(C) + \sum_{i=1}^{k} \nu(T_i),$$

which is a contradiction by Lemma 2.5. As $|V(C)|$ is even and $\nu(C) = \frac{|V(C)|}{2}$, then all maximum matchings in C are perfect matching. Hence, $x \notin V(C)$, otherwise would be saturated by $M \cap E(C)$ and consequently saturated by M. That is, $x \in V(T_s)$ for some $s \in \{1, \dots, k\}$. Since $M \cap E(T_s) \in \mathcal{M}(T_s)$, then by Lemma 2.2 $x \in Supp(T_s)$, that is, $x \in Supp(G-C) = \bigcup\limits_{i=1}^{k} Supp(T_i)$.

Now, given $x \in Supp(G-C)$, we will obtain $M \in \mathcal{M}(G)$ such that M does not saturate x. Consider $G-C = \bigcup\limits_{i=1}^{k} T_i$, where T_i is a connected component of $G-C$. Note that $Supp(G-C) = \bigcup\limits_{i=1}^{k} Supp(T_i)$. Thus, there is $s \in \{1, \dots, k\}$ such that $x \in Supp(T_s)$.

Since $x \in Supp(T_s)$, then by Lemma 2.2 there is $M_s \in \mathcal{M}(T_s)$ such that M_s does not saturate x. Let $M_i \in \mathcal{M}(T_i)$ with $i \in \{1, \dots, k\} - \{s\}$ and $M_c \in \mathcal{M}(C)$. Define $M = M_c \cup \left(\bigcup\limits_{i=1}^{k} M_i \right)$. Note that M is a matching in G and M does not saturate x. Moreover, by Lemma 2.5 we conclude that $M \in \mathcal{M}(G)$.

(ii) Using Lemma 4.5 we conclude that $N(Supp(G-C)) \cap V(C) = \emptyset$. Thus, $N(EG(G)) = N(Supp(G-C)) = Core(G-C)$. Moreover, by Lemma 2.1 we have that $N(EG(G)) \cap EG(G) = N(Supp(G-C)) \cap Supp(G-C) = \emptyset$. Therefore,

$$\begin{aligned} R(G) &= N(EG(G)) - EG(G) \\ &= N(Supp(G-C)) - Supp(G-C) \\ &= Core(G-C). \end{aligned}$$

(iii) Just use the items (i) and (ii).

□

Lemma 4.7. *Let G be a unicyclic graph of Type II and $G\{v\}$ a pendant tree. Let $u \in V(G)$ and $M \in \mathcal{M}(G)$. If $u \in Core(G\{v\}) \cup V(\mathcal{G}_N(G\{v\}))$, then M saturates u.*

Proof. Let C the cycle of G. Suppose there is $M \in \mathcal{M}(G)$ such that M does not saturate u. Note that $M \cap E(G\{v\})$ is a matching of $G\{v\}$ and does not saturate u, thus, by Lemma 2.2 we conclude that $M \cap E(G\{v\}) \notin \mathcal{M}(G\{v\})$. Hence, using Lemma 4.3 we conclude that $M \notin \mathcal{M}(G)$, which is a contradiction. □

Theorem 4.8. *Let G be a unicyclic graph of Type II and C its cycle. If $|V(C)|$ is odd, then*

(i) $EG(G) = \bigcup_{v \in V(C)} Supp(G\{v\})$

(ii) $R(G) = \bigcup_{v \in V(C)} Core(G\{v\})$

(iii) $S(G) = \bigcup_{v \in V(C)} V(\mathcal{G}_N(G\{v\}))$

Proof. (i) Given $x \in EG(G)$. Thus, there is $M \in \mathcal{M}(G)$ such that M does not saturate x. Then by Lemma 4.7 we conclude that $x \in Supp(G\{v\})$ for some $v \in V(C)$.

Given $x \in Supp(G\{v\})$ for some $v \in V(C)$. We will obtain $M \in \mathcal{M}(G)$ such that M does not saturate x. Suppose first $x = v$. Since $|V(C)|$ is odd, then there is $M_c \in \mathcal{M}(C)$ such that M_c does not saturate v. Let $M_w \in \mathcal{M}(G\{w\})$ such that M_w does not saturate w for all $w \in V(C)$ (notice that M_w exists for Lemma 2.2). Define $M = M_c \cup \left(\bigcup_{w \in V(C)} M_w \right)$. Consider $G - C = \bigcup_{i=1}^{k} T_i$, where T_i is a connected component of $G - C$. Note that M is a matching in G does not saturate v and for all $w \in V(C)$ we have that $\nu(G\{w\}) = \nu(G\{w\} - w)$ (see Lemma 3.3). Moreover, we have that

$$\begin{aligned} |M| &= |M_c| + \sum_{w \in V(C)} |M_w| \\ &= \nu(C) + \sum_{w \in V(C)} \nu(G\{w\}) \\ &= \nu(C) + \sum_{w \in V(C)} \nu(G\{w\} - w) \\ &= \nu(C) + \sum_{i=1}^{k} \nu(T_i) = \nu(G). \end{aligned}$$

Then $M \in \mathcal{M}(G)$. Therefore, $x \in EG(G)$.

Now, suppose $x \neq v$. Let $M_v \in \mathcal{M}(G\{v\})$ and $M_c \in \mathcal{M}(C)$ such that M_v does not saturate x and M_c does not saturate v (notice that M_v by Lemma 2.2). Consider $M_u \in \mathcal{M}(G\{u\})$ such that M does not saturate u for all $u \in V(C) - \{v\}$. Define $M = M_c \cup \left(\bigcup_{u \in V(C)} M_u \right)$. Similarly we conclude that $M \in \mathcal{M}(G)$ and does not saturate x. Hence, $x \in EG(G)$.

(ii) Note that $N(EG(G)) = V(C) \cup \left(\bigcup_{v \in V(C)} Core(G\{v\}) \right)$ and $N(EG(G)) \cap EG(G) = V(C)$. Therefore,

$$\begin{aligned} R(G) &= N(EG(G)) - EG(G) \\ &= N\left(\bigcup_{v \in V(C)} Supp(G\{v\}) \right) - \bigcup_{v \in V(C)} Supp(G\{v\}) \\ &= \bigcup_{v \in V(C)} Core(G\{v\}). \end{aligned}$$

(iii) Just use the items (i) and (ii).

□

In this example, we utilize the Theorem 4.8 to obtain the Gallai-Edmonds decomposition of the unicyclic graph G in Figure 3. Consider C the cycle of G. Analyzing the entries of the vectors of the basis of the $\mathcal{N}(G\{a\})$, $\mathcal{N}(G\{b\})$, $\mathcal{N}(G\{c\})$, $\mathcal{N}(G\{d\})$ and $\mathcal{N}(G\{e\})$ we conclude that support, core and N-vertices set of the pendant trees of G are given in Table 1.

Support	Core	N-vertices
$Supp(G\{a\}) = \{g, h, a, l, t\}$	$Core(G\{a\}) = \{f, m\}$	$V(\mathcal{F}_N(G\{a\})) = \{i, j\}$
$Supp(G\{b\}) = \{b\}$	$Core(G\{b\}) = \emptyset$	$V(\mathcal{F}_N(G\{b\})) = \emptyset$
$Supp(G\{c\}) = \{c, n, p, q\}$	$Core(G\{c\}) = \{o\}$	$V(\mathcal{F}_N(G\{c\})) = \{v, r, s, u\}$
$Supp(G\{d\}) = \{d\}$	$Core(G\{d\}) = \emptyset$	$V(\mathcal{F}_N(G\{d\})) = \emptyset$
$Supp(G\{e\}) = \{e\}$	$Core(G\{e\}) = \emptyset$	$V(\mathcal{F}_N(G\{e\})) = \emptyset$

TABLE 1. Support, Core and N-vertices of the pendant trees of G.

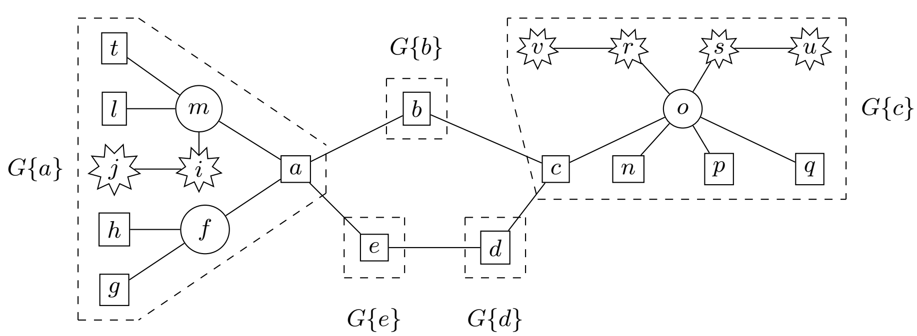

FIGURE 3. Unicyclic graph of Type II and its pendant trees.

Note that G is a unicyclic graph of Type II, because $a \in Supp(G\{a\})$, $b \in Supp(G\{b\})$, $c \in Supp(G\{c\})$, $d \in Supp(G\{d\})$ and $e \in Supp(G\{e\})$. Moreover, $|V(C)|$ is odd. Therefore, by Theorem 4.8 the Gallai-Edmonds decomposition of G is given by:

$$\begin{aligned} EG(G) &= \bigcup_{v \in V(C)} Supp(G\{v\}) = \{a, b, c, d, e, g, h, l, t, n, p, q\}, \\ R(G) &= \bigcup_{v \in V(C)} Core(G\{v\}) = \{f, m, o\}, \\ S(G) &= \bigcup_{v \in V(C)} V(\mathcal{G}_N(G\{v\})) = \{v, r, s, u, i, j\}. \end{aligned}$$

5. Concluding Remark

We have provided in this paper a way to obtain the Gallai-Edmonds decomposition of unicyclic graphs from the null space of its adjacency matrix.

Similarly to the Gallai-Edmonds decomposition, the Zito decomposition [19] is also a partition of the vertex set. The difference is that Zito decomposition is defined through certain properties of the maximum independent sets. In trees and C_{4k}-free bipartite graphs all three decompositions coincide, that is, form the same partition of the vertex set [2, 11, 12]. For graphs in general, this is not the case (consider, for example, the pan graph of order 6, that is, the graph obtained by joining the cycle graph C_6 to the complete graph K_1 with an edge).

In view of this fact, the following question naturally arises: Is there a relationship between Zito decomposition of unicyclic graphs and null decomposition of its subtrees?

Acknowledgments

Work supported by MATHAMSUD grant 18-MATH-01. Daniel A. Jaume and Gonzalo Molina were parcially supported by PROICO 03-0918 UNSL. V. Trevisan acknowledges partial support of CNPq grants 409746/2016-9 and 303334/2016-9, CAPES-Print 88887.467572/2019-00 and FAPERGS (Proj. PqG 17/2551-0001).

References

[1] R. Aharoni and R. Ziv. LP duality in infinite hypergraphs. *Journal of Combinatorial Theory, Series B* 50.1 (1990): 82-92.

[2] L. E. Allem, D. A. Jaume, G. Molina, M. M. Toledo and V. Trevisan, Independence and matching numbers of unicyclic graphs from null space. *Computational and Applied Mathematics* 39, 2 (2020), 64.

[3] L. E. Allem, D. A. Jaume, G. Molina, M. M. Toledo and V. Trevisan, Null decomposition of unicyclic graphs. *arXiv preprint arXiv*: 1907.08618 (2019).

[4] M. Bartha and E. Gombás. A structure theorem for maximum internal matchings in graphs. *Information Processing Letters* 40.6 (1991): 289-294.

[5] F. Bry and M. L. Vergnas. The Edmonds-Gallai decomposition for matchings in locally finite graphs.*Combinatorica* 3 (1982): 229-235.

[6] R. Cymer. Gallai-Edmonds decomposition as a pruning technique. *Central European Journal of Operations Research* 23.1 (2015): 149-185.

[7] J. Edmonds, Paths, trees, and flowers. *Canadian Journal of mathematics* 17, 3 (1965), 449467.

[8] Q. Feng, G. Tan, S. Zhu, B. Fu and J. Wang. New algorithms for edge induced König-egerváry subgraph based on gallai-edmonds decomposition. 29th International Symposium on Algorithms and Computation (ISAAC 2018). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018.

[9] T. Gallai, Kritische Graphen II. *Magyar Tudományos Akadémia; Matematikai Kutató Intézetének Közleményei.* 8:373395, (1963).

[10] T. Gallai, Maximale systeme unabhangiger kanten. *Magyar Tudományos Akadémia; Matematikai Kutató Intézetének Közleményei.* 9 (1964), 401413.

[11] D. A. Jaume and G. Molina. Null Decomposition of Trees. *Discrete Mathematics*, 341:836-850, 2018.

[12] D. A. Jaume, G. Molina, and A. Pastine. Null decomposition of bipartite graphs without cycles of length 0 modulo 4, *Linear Algebra and its Applications*, in press, 2020. https://doi.org/10.1016/j.laa.2020.03.030

[13] B. Ji and Y. Sang. Throughput characterization of node-based scheduling in multihop wireless networks: a novel application of the Gallai-Edmonds structure theorem. *Proceedings of the 17th ACM International Symposium on Mobile Ad Hoc Networking and Computing* 2016.

[14] Y. Liu, M. Lei, and X. Su. Fractional Gallai-Edmonds decomposition and maximal graphs on fractional matching number. *Journal of Combinatorial Optimization*: 1-10, 2020.

[15] L. Lovász, and M. D. Plummer, M. D. Matching theory, vol. 367. *American Mathematical Soc.*, 2009.

[16] B. Spille and L. Szegő. A Gallai-Edmonds-type structure theorem for path-matchings. *Journal of graph theory* 46.2 (2004): 93-102.

[17] K. Steffens. Maximal tight sets and the Edmonds-Gallai decomposition for matchings. *Combinatorica* 5.4 (1985): 359-365.

[18] M. Stehlík. A hypergraph version of the Gallai-Edmonds Theorem. *Electronic Notes in Discrete Mathematics* 28 (2007): 387-391.

[19] J. Zito, The structure and maximum number of maximum independent sets in trees. *Journal of Graph Theory* 15, 2 (1991), 207221. 106

E-mail address: `emilio.allem@ufrgs.br`

UFRGS - Universidade Federal do Rio Grande do Sul, Instituto de Matemática, Porto Alegre, Brazil

E-mail address: `djaume@unsl.edu.ar`

Instituto de Matemáticas Aplicadas de San luis UNSL-CONCIET, Universidad Nacional de San Luis, Departamento de Matemáticas, San Luis, Argentina

E-mail address: `lgmolina@unsl.edu.ar`

Instituto de Matemáticas Aplicadas de San luis UNSL-CONCIET, Universidad Nacional de San Luis, Departamento de Matemáticas, San Luis, Argentina

E-mail address: `maikon.toledo@ufrgs.br`

UFRGS - Universidade Federal do Rio Grande do Sul, Instituto de Matemática, Porto Alegre, Brazil

E-mail address: `trevisan@mat.ufrgs.br`

UFRGS - Universidade Federal do Rio Grande do Sul, Instituto de Matemática, Porto Alegre, Brazil and Department of Mathematics and Applications, University of Naples Federico II, Italy