ON TREES WITH A UNIQUE MAXIMUM INDEPENDENT SET AND THEIR LINEAR PROPERTIES

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Abstract. Trees with a unique maximum independent set encode the maximum matching structure in every tree. In this work we study some of their linear properties and give two graph operations, stellare and S-coalescence, which allow to build all trees with a unique maximum independent set. The null space structure of any tree can be understood in terms of these graph operations.

1. Introduction

The trees with a unique maximum independent set were first characterized by Hopkins and Staton (1985) in [4]. Sander and Sander (2005), see [8], gave another characterization in terms of the FOX algorithm. In [5] they were characterized using linear algebra and the null decomposition of trees. Furthermore, in [5] it was proved that every tree can be decomposed in a forest of subtrees with a unique (perfect) maximum matching, and a forest of subtrees with a unique maximum independent set. This last forest encode the main part of the maximum matching structure of a tree.

In this work we study many properties of the trees with a unique maximum independent set. We call them independent trees. Furthermore, we describe all this properties in term of its atom forest (spanning forest of strong unique independence subtrees), see Section 4. We also give two graph operations, stellare and S-coalescence, which allows to build every tree from very simple subtrees.

This paper is organized as follows. In Section 2, we work with atoms trees. They are independent trees with the property that the complement of its unique maximum independent set is also an independent set of the tree, see [4]. In Section 3 the trees with a unique independent set are study in depth. Finally, in Section 4 we give the two graph operations which allow building easily every independent tree.

Let us now introduce some notation required later on. All graphs in this work are labeled (even when we do not write the labels), finite, undirected and with neither loops nor multiple edges. Let G be a graph. By $V(G)$ we denoted its set of vertices, and $v(G) := |V(G)|$. Following Bapat, we use uppercase letters for sets

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and lowercase for their cardinalities, see [1]. Similarly, By $E(G)$ we denote its set of edges, and by $e(G)$ we denote the size of G. The neighborhood of $v \in V(G)$, denoted by $N(v)$, is the set $\{u \in V(G) : u \sim v\}$, where $u \sim v$ means that $\{u, v\} \in E(G)$. The neighborhood of a subset S of vertices of G is $N(S) := \bigcup_{v \in S} N(v)$. The closed neighborhood of v, denoted by $N[v]$, is the set of vertices $\{v\} \cup N(v)$. The closed neighborhood of S, denoted by $N[S]$, is the set $S \cup N(S)$. For all graph-theoretic notions not defined here, the reader is referred to [3]. Let $u, v \in V(G)$. By $G + \{u, v\}$ we denote the graph obtained by adding the edge $\{u, v\}$ to $E(G)$. Let $e \in E(G)$. By $G - e$ we denote the graph obtained by removing the edge e from G, thus $E(G - e) = E(G) - \{e\}$. By $\deg(v)$, we denote the degree of a vertex v of a graph G, that is, $\deg(v) = |N(v)|$. A vertex v of a graph G is a pendant vertex if $\deg(v) = 1$. We use $[n]$ instead of $\{1, \ldots, n\}$.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. By $\mathbb{R}^G$ we denote the vector space of all functions from $V(G)$ to $\mathbb{R}$. Let $\vec{x} \in \mathbb{R}^G$ and $v \in V(G)$. We write $\vec{x}_v$ instead of $\vec{x}(v)$. By e_v we denote the standard basis vector at v, i.e. $e_v(u) = 1$ if $v = u$ and $e_v(u) = 0$ if $v \neq u$. With θ we denote the null vector of $\mathbb{R}^G$. For all linear algebra-theoretic notions not defined here, the reader is referred to [6].

2. Atom trees and null decomposition of trees

The null space of a graph G, denoted by $\mathcal{N}(G)$, is the null space of its adjacency matrix. Thus, $\mathcal{N}(G) = \mathcal{N}(A(G))$. The nullity of G is the nullity of its adjacency matrix: $\text{null}(G) = \text{null}(A(G))$.

The null space of any graph decomposes the vertices of the graph in three sets: the support, the core, and the invertible part, see [5].

Definition 2.1. *Let G be a graph. The* ***support*** *of G, denoted by* $\text{Supp}(G)$*, is the set of vertices of G*

$$\{v \in V(G) : \exists \vec{x} \in \mathcal{N}(G), \text{ such that } \vec{x}_v \neq 0\}.$$

The ***core*** *of G, denoted by* $\text{Core}(G)$*, is the set of vertices* $N(\text{Supp}(G)) - \text{Supp}(G)$*. The* ***invertible*** *part of G, denoted by* $\text{Inv}(G)$*, is* $V(G) - N[\text{Supp}(G)]$.

The vertices in $\text{Supp}(G)$ are called the supported vertices of T. Following Bapat, see [1], we write $\text{supp}(T)$ instead of $|\text{Supp}(T)|$. The vertices in $\text{Core}(G)$ are called the core vertices of T. We write $\text{core}(T)$ instead of $|\text{Core}(T)|$. The vertices in $\text{Inv}(G)$ are called the invertible vertices of T. We write $\text{inv}(T)$ instead of $|\text{Inv}(T)|$.

The next lemma is a rephrasing of Lemma 2.2 of [5].

Lemma 2.2 ([5])**.** *Let G be a graph. If $S \subset \text{Supp}(G)$, then there exists $\vec{x} \in \mathcal{N}(G)$ such that for all $v \in S$, $\vec{x}_v \neq 0$.*

An independent set I of a graph G is a set of vertices of G pairwise non-neighbors. The independence number of G, denoted by $\alpha(G)$, is the maximum cardinality of the independent sets of G. The set of all maximum independent sets of a graph G is denoted by $\mathcal{I}(G)$, and its cardinality by $a(G)$, i.e. $a(G) = |\mathcal{I}(G)|$.

Lemma 2.3 ([5], Lemma 2.6)**.** *If T is a tree, then* $\text{Supp}(T)$ *is an independent set of T.*

The following lemma is a slightly generalized version of Lemma 3.4 in [5].

Lemma 2.4. *Let T be tree. If $v \in \mathrm{Core}(T)$, then $|N(v) \cap \mathrm{Supp}(T)| \geqslant 2$.*

Proof. Clearly $|N(v) \cap \mathrm{Supp}(S)| > 0$. By Lemma 2.2 there exists $\vec{x} \in \mathcal{N}(T)$ such that for all $u \in \mathrm{Supp}(T)$, we have $\vec{x}_u \neq 0$. Since $\sum_{w \sim v} \vec{x}_w = 0$, there are at least two vertices of $N(v)$ such that its respective coordinates in $\vec{x}$ are nonzero, and this is precisely the assertion of the lemma. ■

The trees with a unique maximum independent set were first characterized by Hopkins and Staton (1985), see [4]. In that work they introduce the notion of **strong maximum independent set**: Let I be a maximum independent set of G, if $I^c := V(G) - I$ is also an independent set of G, then we say that I is a strong maximum independent set of G. Clearly, graphs with a strong maximum independent set are bipartite.

Theorem 2.5. *Let T be a tree. The following statements are equivalent:*

(1) *T has a unique strong maximum independent set.*
(2) *If u and v are pendant vertices of T, then the distance between them even.*
(3) *The tree T is $(\mathrm{Supp}(T), \mathrm{Core}(T))$-bipartite.*

A tree that satisfies any (and hence all of them) of these conditions is called an ***atom*** *tree.*

Proof. By Theorem 3 of [4] statements (1) and (2) are equivalent. Assume that (3) holds. By Lemma 2.4, if v is a pendant vertex of T, then $v \in \mathrm{Supp}(T)$. Hence, (3) implies (2). In order to prove that (2) implies (3) fix u, a pendant vertex of T. Any other vertex of T is in a path between u and another pendant vertex of T, named w. By P we denote the unique path from u to w in T. We define $\vec{x}(u, w)$ as follows:

$$\vec{x}(u,w)_v = \begin{cases} 1 & \text{if } d(u,v) \equiv_4 0 \text{ and } v \in V(P), \\ -1 & \text{if } d(u,v) \equiv_4 2 \text{ and } v \in V(P), \\ 0 & \text{otherwise.} \end{cases}$$

If $\vec{x}(u, w) \notin \mathcal{N}(T)$, then there exist $z \in V(T)$ such that $d(u, z) \equiv_2 1$ and

$$\sum_{v \in N(z)} \vec{x}(u,w)_v \neq 0.$$

Therefore, there exists a unique $t_1 \in V(P) \cap N(z)$ such that $\vec{x}(u, w)_{t_1} \neq 0$. Since $d(u, z) \equiv_2 1$, z is not a pendant vertex of T. Therefore there must exist a $t_2 \in N(z) \cap (V(T) - V(P))$. Redefine $\vec{x}(u, w)_{t_2} = -\vec{x}(u, w)_{t_1}$, and P as $P + \{t_1, z\} + \{z, t_1\}$. If this new $\vec{x}(u, w)$ is not in $\mathcal{N}(T)$ repeat the former process. After a finite number of rounds, we arrive to a vector in the null space of T. Hence, $\{v \in V(T) : d(u, v) \equiv_2 0\} \subset \mathrm{Supp}(T)$.

Since T is a tree, the sets $V_0 = \{v \in V(T) : d(u, v) \equiv_2 0\}$ and $V_1 = \{v \in V(T) : d(u, v) \equiv_2 1\}$ form a bipartition of T. Note that $V_1 = N(V_0) \subset \mathrm{Supp}(T)$. Thus $V_1 \subset \mathrm{Core}(T)$. Therefore,

$$V(T) = V_0 \cup V_1 \subset \mathrm{Supp}(T) \cup \mathrm{Core}(T) \subset V(T),$$

Hence, $\mathrm{Supp}(T) = V_1$ and $\mathrm{Core}(T) = V_2$. ■

We usually write $\mathfrak{A}$ for atom trees. In [4] the atoms trees are called strong unique independence trees. A very similar notion can be traced in the work of Sander and Sander, see [8], in terms of the FOX algorithm.

The null decomposition of trees breaks any tree in two forests. A forest of trees with a unique maximum matching (a perfect matching), and a forest of trees with a unique maximum independent set. The trees with a unique maximum matching has a non singular adjacency matrix, see [2], [5], or [1]. They are called **invertible** or **matching** trees.

Let G be a graph. Given $U \subset V(G)$, the subgraph of G induced by U is denoted by $G\langle U\rangle$. The set of all connected components of a graph G is denoted by $\mathcal{K}(G)$.

Let T be a tree. We set $\mathcal{F}_{\text{indep}}(T) := T\langle N[\text{Supp}(T)]\rangle$, and $\mathcal{F}_{\text{match}}(T) := T\langle \text{Inv}(T)\rangle$. The forest $\mathcal{F}_{\text{indep}}(T)$ is called the **independence forest** of T. The forest $\mathcal{F}_{\text{match}}(T)$ is called the **matching forest** of T. The empty set is simultaneously an independence forest of T and a matching forest of T. These notions were introduced in [5] under the name of S-set and N-set of T.

Theorem 2.6 ([5]). *Let T be a tree. The independence forest of T is a forest of independent trees and the matching forest of T is a forest of matching trees.*

Definition 2.7 ([5]). *The **connection edges** of a tree T, denoted by $CE(T)$, is the set of all the edges between a core vertex and an invertible vertex:*

$$\{\{u, v\} \in E(T) : u \in \text{Core}(T) \text{ and } v \in \text{Inv}(T)\}.$$

Let $\mathcal{F}_{\text{null}}(T) := \mathcal{F}_{\text{indep}}(T) \cup \mathcal{F}_{\text{match}}(T)$. This forest associated with T is called the **null forest** of T. The following result is implicit in [5].

Theorem 2.8 ([5]). *If T is a tree, then $E(T - \mathcal{F}_{null}(T)) = CE(T)$.*

We use the same symbol for a forest and for the set of all its connected components: $H \in \mathcal{F}$ means $H \in \mathcal{K}(\mathcal{F})$, where $\mathcal{F}$ is a forest.

Theorem 2.9 ([5]). *If T is a tree, then*

$$\text{Supp}(T) = \bigcup_{S \in \mathcal{F}_{indep}(T)} \text{Supp}(S),$$

$$\text{Core}(T) = \bigcup_{S \in \mathcal{F}_{indep}(T)} \text{Core}(S),$$

$$\text{Inv}(T) = \bigcup_{N \in \mathcal{F}_{match}(T)} \text{Inv}(N).$$

The row space of a graph G, denoted by $\mathcal{R}(G)$, is the row space of its adjacency matrix. Thus, $\mathcal{R}(G) = \mathcal{R}(A(G))$. The rank of a graph G is the rank of its adjacency matrix: $\text{rk}(G) = \text{rk}(A(G))$. By $\nu(G)$ we denote the matching number of G, i.e. cardinality of a maximum matching of G. By $\mathcal{M}(G)$ we denote the set of all maximum matchings in G. The number of maximum matchings in G is denoted $m(G)$. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The independence number $\alpha(G)$ of a graph G and its domination number $\gamma(G)$ are related by $\gamma(G) \leqslant \alpha(G)$.

Theorem 2.10 ([5])**.** *If T is a tree, then*

$$\begin{aligned}
\mathrm{null}(T) &= \mathrm{supp}(T) - \mathrm{core}(T),\\
\mathrm{rk}(T) &= 2\,\mathrm{core}(T) + \mathrm{inv}(T),\\
\nu(T) &= \mathrm{core}(T) + \frac{\mathrm{inv}(T)}{2},\\
m(T) &= \prod_{S \in \mathcal{F}_{indep}(T)} m(S),\\
\alpha(T) &= \mathrm{supp}(T) + \frac{\mathrm{inv}(T)}{2},\\
a(T) &= \prod_{N \in \mathcal{F}_{match}(T)} a(N).
\end{aligned}$$

Corollary 2.11. *Let T be a tree. Then $\nu(T) = \alpha(T) - \mathrm{null}(T)$.*

3. Trees with a unique maximum independent set

In this section we study some linear properties of the trees with a unique maximum independent set.

Theorem 3.1 ([4], Theorem 6, and [5], Section 3 and Corollary 4.18)**.** *Let T be a tree. The following statements are equivalent:*

(1) $a(T) = 1$,
(2) $\mathrm{Supp}(T)$ *is the unique maximum independent set of T,*
(3) $N[\mathrm{Supp}(T)] = V(T)$,
(4) *T has a spanning forest $\mathcal{F}$ such that each component of $\mathcal{F}$ is an atom tree, and each edge in $E(T) - E(\mathcal{F})$ joins two core vertices of T.*

A tree that satisfies any (and hence all of them) of these conditions is called an ***independent tree****.*

An independent tree and its spanning forest of atom trees is shown in Figure 1. In [5] the independent trees are called S-trees, because the spotlight was on the linear structure of the null space of the adjacency matrix. The independent trees can also be characterized via FOX algorithm, see [8].

From this result, an independent tree can be built from a forest of atom trees. In the next section we give two graph-theoretical operations, with linear algebra flavor, that allow us to build every independent tree.

Definition 3.2. *The* ***bond edges*** *of a tree T, denoted by $BE(T)$, is the set of all the edges between core vertices:*

$$\{\{u, v\} \in E(T) \,:\, u, v \in \mathrm{Core}(T)\}.$$

We prove that for each independent tree there exist just one spanning forest of atoms: $\mathcal{F}_{\mathrm{atom}}(T) := \mathcal{K}(T - BE(T))$, see Theorem 3.5. They also are important for the maximum matching structure of any tree. See Corollary 3.8.

Let T be an independent tree. Let $\mathcal{F}$ be a spanning forest of atoms of T such that $E(T) - E(\mathcal{F}) \subset BE(T)$, and $\mathfrak{A} \in \mathcal{F}$. Then

(1) $\deg_T(v) = \deg_{\mathfrak{A}}(v)$, for all $v \in \mathrm{Supp}(T) \cap V(\mathfrak{A})$,
(2) if v is a pendant vertex of $\mathfrak{A}$, then v is a pendant vertex of T.

We need the following notation introduced in [5]. Given a graph G, let $\vec{x}$ be a vector of $\mathbb{R}^G$. Let H be a subgraph of G. The vector obtained when we restrict $\vec{x}$ to the coordinates (vertices) associated with H is denoted by $\vec{x}\downharpoonright^G_H$. By $\vec{y}\upharpoonright^G_H$ we denote the lift of vector $\vec{y} \in \mathbb{R}^H$ to a vector of $\mathbb{R}^G$: for any $u \in V(G) - V(H)$, $(\vec{y}\upharpoonright^G_H)_u := 0$, and for any $u \in V(H)$, $(\vec{y}\upharpoonright^G_H)_u := \vec{y}_u$.

Theorem 3.3. *Let T be an independent tree and $\mathcal{F}$ be a spanning forest of atoms of T such that $E(T) - E(\mathcal{F}) \subset BE(T)$. Then the following statements hold.*

(1) *If $\mathfrak{A} \in \mathcal{F}$, then*
   (a) $\mathrm{Supp}(\mathfrak{A}) = \mathrm{Supp}(T) \cap V(\mathfrak{A})$,
   (b) $\mathrm{Core}(\mathfrak{A}) = \mathrm{Core}(T) \cap V(\mathfrak{A})$.
(2) $\mathrm{Supp}(T) = \bigcup_{\mathfrak{A}\in\mathcal{F}} \mathrm{Supp}(\mathfrak{A})$.
(3) $\mathrm{Core}(T) = \bigcup_{\mathfrak{A}\in\mathcal{F}} \mathrm{Core}(\mathfrak{A})$.

Proof. By Lemma 2.2, we can take $\vec{x} \in \mathcal{N}(T)$ such that $\vec{x}_v \neq 0$ if and only if $v \in \mathrm{Supp}(T)$.

Claim 1: $\vec{x}\downharpoonright^T_{\mathfrak{A}} \in \mathcal{N}(\mathfrak{A})$.

Therefore, $\mathrm{Supp}(T) \cap V(\mathfrak{A}) \subset \mathrm{Supp}(\mathfrak{A})$. Hence, $\mathrm{Core}(\mathfrak{A}) \subset \mathrm{Core}(T) \cap V(\mathfrak{A})$.

Claim 2: $\mathrm{Core}(T) \cap V(\mathfrak{A}) \subset \mathrm{Core}(\mathfrak{A})$.

Since

$$(\mathrm{Supp}(T) \cap V(\mathfrak{A})) \,\dot\cup\, (\mathrm{Core}(T) \cap V(\mathfrak{A})) = V(\mathfrak{A}) = \mathrm{Supp}(\mathfrak{A}) \,\dot\cup\, \mathrm{Core}(\mathfrak{A}),$$

we can conclude that $\mathrm{Supp}(T) \cap V(\mathfrak{A}) = \mathrm{Supp}(\mathfrak{A})$.

Proof of the Claim 1. We proof the claim in a coordinatewise fashion. If $v \in \mathrm{Supp}(\mathfrak{A})$, then

$$\left(A(\mathfrak{A})\,\vec{x}\downharpoonright^T_{\mathfrak{A}}\right)_v = \left(A(T)\vec{x}\right)_v = 0.$$

If $v \in \mathrm{Core}(\mathfrak{A})$, then

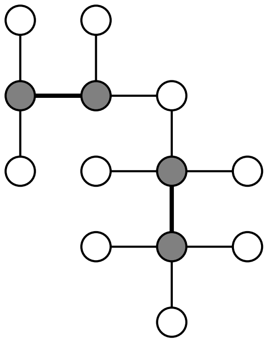
$$0 = \left(A(T)\vec{x}\right)_v = \left(A(\mathfrak{A})\,\vec{x}\downharpoonright^T_{\mathfrak{A}}\right)_v + \sum_{\substack{u\sim v\\ u\notin V(\mathfrak{A})}} \vec{x}_u.$$

Since, if $u \sim v$ and $u \notin V(\mathfrak{A})$, then $\{u, v\} \in E(T) - E(\mathcal{F}) \subset BE(T)$. Hence, $\vec{x}_u = 0$, if $u \sim v$ and $u \notin V(\mathfrak{A})$. Therefore, $\left(A(\mathfrak{A})\,\vec{x}\downharpoonright^T_{\mathfrak{A}}\right)_v = 0$. This proves the claim 1.

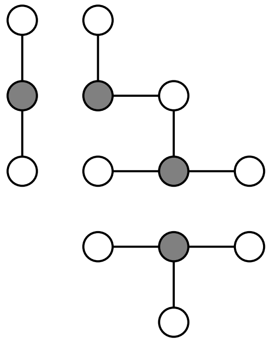
Proof of the Claim 2. If $v \in \mathrm{Core}(T) \cap V(\mathfrak{A})$, then there exists $u \in \mathrm{Supp}(T)$ such that $u \sim v$. Therefore, since $E(T) - E(\mathcal{F}) \subset BE(T)$, the vertex u is a vertex of $\mathfrak{A}$. Thus, $u \in \mathrm{Supp}(T) \cap V(\mathfrak{A}) \subset \mathrm{Supp}(\mathfrak{A})$. Hence, since $v \in V(\mathfrak{A})$, we conclude $v \in \mathrm{Core}(\mathfrak{A})$.

The other two statements are now obvious. ∎

Corollary 3.4. *Let T be an independent tree and $\mathcal{F}$ be an spanning forest of atoms of T such that $E(T) - E(\mathcal{F}) \subset BE(T)$. Let $e \in E(T) - E(\mathcal{F})$ and $\{T_1, T_2\} := \mathcal{K}(T - e)$. Then $\mathcal{F}_1 := \mathcal{F}\langle(T_1)\rangle$ is a spanning forest of atoms of T_1 such that $E(T_1) - E(\mathcal{F}_1) \subset BE(T_1)$, and $\mathcal{F}_2 := \mathcal{F}\langle V(T_2)\rangle$ is a spanning forest of atoms of T_2 such that $E(T_2) - E(\mathcal{F}_2) \subset BE(T_2)$. Furthermore $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$.*



(A) An independent tree T



(B) Its atom forest $\mathcal{F}_{\text{atom}}(T)$

FIGURE 1. An independent tree T and its atom forest $\mathcal{F}_{\text{atom}}(T)$.

The next results says that the forest decomposition into atoms of independent trees is unique.

Theorem 3.5. *Let T be an independent tree. If $\mathcal{F}$ is a spanning forest of atoms of T such that $E(T) - E(\mathcal{F}) \subset BE(T)$, then $\mathcal{F} = \mathcal{F}_{atom}(T)$.*

Proof. By induction on $|BE(T)|$ and Corollary 3.4. ∎

Theorem 3.6 ([5])**.** *If T is an independent tree and $M \in \mathcal{M}(T)$, then each edge of M matches one supported vertex with one core vertex of T.*

The next result is a direct consequence of 3.6.

Corollary 3.7. *Let T be an independent tree. The following statements are true (the proofs are similar to the ones in* [5]*).*

(1) *If $M \in \mathcal{M}(T)$, then for each $\mathfrak{A} \in \mathcal{F}_{atom}(T)$ we have $M \cap E(\mathfrak{A}) \in \mathcal{M}(\mathfrak{A})$.*
(2) *For each $\mathfrak{A} \in \mathcal{F}_{atom}(T)$, let $M_{\mathfrak{A}} \in \mathcal{M}(\mathfrak{A})$. Then*

$$\bigcup_{\mathfrak{A} \in \mathcal{F}_{atom}(T)} M_{\mathfrak{A}} \in \mathcal{M}(T).$$

Corollary 3.8. *If T is an independent tree, then*

$$m(T) = \prod_{\mathfrak{A} \in \mathcal{F}_{atom}(T)} m(\mathfrak{A}).$$

The null decomposition of trees says, amongst other things, that the "variations" in the maximum matching structure of any tree are determined by its independent forest, while the "variations" in the maximum independent structure of any tree are determined by its matching forest. Parts of following theorem are implicit in [5].

Theorem 3.9. *Let T be a tree. The following statements are true:*

(1) *If* $M \in \mathcal{M}(T)$*, then for each* $S \in \mathcal{F}_{indep}(T)$ *we have* $M \cap E(S) \in \mathcal{M}(S)$*, and for each* $H \in \mathcal{F}_{match}(T)$ *we have* $M \cap E(H)$ *is the perfect matching of* H*.*

(2) *For each* $H \in \mathcal{F}_{null}(T)$*, let* $M_H \in \mathcal{M}(H)$*. Then*

$$\bigcup_{H \in \mathcal{F}_{null}(T)} M_H \in \mathcal{M}(T).$$

(3) *If* $I \in \mathcal{I}(T)$*, then for each* $N \in \mathcal{F}_{match}(T)$ *we have* $I \cap V(N) \in \mathcal{I}(N)$*, and for each* $S \in \mathcal{F}_{indep}(T)$ *we have* $I \cap V(S) = \mathrm{Supp}(S)$ *is the unique maximum independent set of* S*.*

(4) *For each* $H \in \mathcal{F}_{null}(T)$*, let* $I_H \in \mathcal{I}(H)$*. Then*

$$\bigcup_{H \in \mathcal{F}_{null}(T)} I_H \in \mathcal{I}(T).$$

Proof. The proofs of statements (1) and (2) are implicit in [5]. By Theorem 3.3 we know that $\mathrm{Supp}(T) = \bigcup_{S \in \mathcal{F}_{\mathrm{indep}}(T)} \mathrm{Supp}(S)$, and that for all $S \in \mathcal{F}_{\mathrm{indep}}(T)$, $\mathrm{Supp}(S) = \mathrm{Supp}(T) \cap V(S)$. The vertices of T satisfy $V(T) = \bigcup_{S \in \mathcal{F}_{\mathrm{indep}}(T)} V(S) \cup \bigcup_{N \in \mathcal{F}_{\mathrm{match}}(T)} V(N)$, where at most one of the "big" unions can be empty. If I is a maximum independent set of a tree T, then

$$\begin{aligned}
\mathrm{supp}(T) + \frac{\mathrm{inv}(T)}{2} &= |I| \\
&= \sum_{S \in \mathcal{F}_{\mathrm{indep}}(T)} |I \cap V(S)| + \sum_{N \in \mathcal{F}_{\mathrm{match}}(T)} |I \cap V(N)| \\
&\leqslant \sum_{S \in \mathcal{F}_{\mathrm{indep}}(T)} \mathrm{supp}(S) + \sum_{N \in \mathcal{F}_{\mathrm{match}}(T)} \frac{v(N)}{2} \\
&= \mathrm{supp}(T) + \frac{\mathrm{inv}(T)}{2},
\end{aligned}$$

Hence, for each $S \in \mathcal{F}_{\mathrm{indep}}(T)$ we have $|I \cap V(S)| = \mathrm{supp}(S)$, and for each $N \in \mathcal{F}_{\mathrm{match}}(T)$, we have $|I \cap V(N)| = v(N)/2$. Hence, $I \cap V(S) = \mathrm{Supp}(S)$, because $I \cap V(S)$ is an independent set of S and S has a unique maximum independent set: $\mathrm{Supp}(S)$. Similarly, for all $N \in \mathcal{F}_{\mathrm{match}}(T)$ we have $I \cap V(N)$ is an independent set of N and $|I \cap V(N)| = \frac{v(N)}{2}$. Therefore, for all $N \in \mathcal{F}_{\mathrm{match}}(T)$ the set $I \cap V(N)$ is a maximum independent set of N.

Statement (4) is a direct consequence of Theorem 2.8. ∎

Corollary 3.10. *Let* T *be a tree. If* $I \in \mathcal{I}(T)$*, then* $I \cap \mathrm{Supp}(T) = \mathrm{Supp}(T)$ *and* $|I \cap \mathrm{Inv}(T)| = \frac{\mathrm{inv}(T)}{2}$*.*

The maximum degree of all the core vertices in an atom tree $\mathfrak{A}$ will be denoted by $\Delta_{\mathrm{core}}(\mathfrak{A})$. It provides a lower bound for the nullity of $\mathfrak{A}$.

Theorem 3.11. *Let* $\mathfrak{A}$ *be an atom tree. Then* $\mathrm{null}(\mathfrak{A}) \geqslant \Delta_{core}(\mathfrak{A}) - 1$*.*

Proof. Let $\mathfrak{A}$ be an atom tree, and $u \in \mathrm{Core}(\mathfrak{A})$ such that $\deg(u) = \Delta_{\mathrm{core}}(\mathfrak{A})$. It is clear that:

$$\begin{aligned}\mathrm{supp}(\mathfrak{A}) &\geqslant \Delta_{\mathrm{core}}(\mathfrak{A}) + \sum_{\substack{v\in \mathrm{Core}(\mathfrak{A})\\ d(v,u)=2}} 1 + \sum_{\substack{v\in \mathrm{Core}(\mathfrak{A})\\ d(v,u)=4}} 1 + \cdots + \sum_{\substack{v\in \mathrm{Core}(\mathfrak{A})\\ d(v,u)=diam(\mathfrak{A})-2}} 1\\ &= \Delta_{\mathrm{core}}(\mathfrak{A}) + (\mathrm{core}(\mathfrak{A}) - 1)\end{aligned}$$

Then, by Theorem 2.10, $\mathrm{null}(\mathfrak{A}) = \mathrm{supp}(\mathfrak{A}) - \mathrm{core}(\mathfrak{A}) \geqslant \Delta_{\mathrm{core}}(\mathfrak{A}) - 1$. ■

For any atom tree $\mathfrak{A}$, its rank is $2\,\mathrm{core}(\mathfrak{A})$. But actually, we can give a basis of its row space.

Definition 3.12. *Let $v \in \mathrm{Core}(T)$, the v-**bouquet** of T, denoted by $R(v)$, is*

$$R(v) := \{u \in \mathrm{Supp}(T) : u \sim v\}.$$

*The v-**bouquet vector**, denoted by $\vec{R}(v)$, is*

$$\vec{R}(v) = \sum_{u\in R(v)} e_u.$$

The “R” is because bouquet is “ramillete” in Spanish.

Lemma 3.13. *Let $\mathfrak{A}$ be an atom tree. The set*

$$\mathcal{B}_{rk}(\mathfrak{A}) := \{e_v, \vec{R}(v) \in \mathbb{R}^{\mathfrak{A}} : v \in \mathrm{Core}(\mathfrak{A})\},$$

is a basis of $\mathcal{R}(\mathfrak{A})$, the row space of $\mathfrak{A}$.

Proof. Let $A_{*\,u}$ be the column of $A(\mathfrak{A})$, the adjacency matrix of $\mathfrak{A}$, associated with the vertex u. If $u \in \mathrm{Supp}(\mathfrak{A})$, then

$$A_{*\,u} = \sum_{\substack{v\in \mathrm{Core}(\mathfrak{A})\\ v\sim u}} e_v.$$

If $u \in \mathrm{Core}(\mathfrak{A})$, then

$$A_{*\,u} = \sum_{\substack{v\in \mathrm{Supp}(\mathfrak{A})\\ v\sim u}} e_v = \vec{R}(u).$$

Hence, $\mathcal{R}(\mathfrak{A}) \subset \langle \mathcal{B}_{rk}(\mathfrak{A})\rangle$. Clearly $\mathcal{B}_{rk}(\mathfrak{A})$ is a set of linearly independent vectors, and $|B_{rk}(\mathfrak{A})| = 2\,\mathrm{core}(\mathfrak{A})$. Therefore, $B_{rk}(\mathfrak{A})$ is a basis of $\mathcal{R}(\mathfrak{A})$. ■

Let G be a graph and H a subgraph of G. Let $A \subset \mathbb{R}^H$ and $B \subset \mathbb{R}^G$. By $A\upharpoonleft^G_H$ we denote the following set of vectors of $\mathbb{R}^G$: $\{\vec{x}\upharpoonleft^G_H : \vec{x} \in A\}$, and by $B\downharpoonleft^G_H$ we denote the following set of vectors of $\mathbb{R}^H$: $\{\vec{y}\downharpoonleft^G_H : \vec{y} \in B\}$.

Theorem 3.14. *Let T be an independent tree. Then*

1. $\mathrm{null}(T) = \sum_{\mathfrak{A}\in\mathcal{F}_{atom}(T)} \mathrm{null}(\mathfrak{A})$,
2. $\mathrm{rk}(T) = \sum_{\mathfrak{A}\in\mathcal{F}_{atom}(T)} \mathrm{rk}(\mathfrak{A})$,
3. $\mathcal{N}(T) = \bigoplus_{\mathfrak{A}\in\mathcal{F}_{atom}(T)} \mathcal{N}(\mathfrak{A})\upharpoonleft^T_{\mathfrak{A}}$, *and*
4. $\mathcal{R}(T) = \bigoplus_{\mathfrak{A}\in\mathcal{F}_{atom}(T)} \mathcal{R}(\mathfrak{A})\upharpoonleft^T_{\mathfrak{A}}$.

Proof. Statements (1) and (2) follow by applying Theorem 2.10 to independent trees and atom trees in particular, taking into account Theorem 3.3.

Proof of statement (3). It is clear that all the subspaces $\mathcal{N}(\mathfrak{A})$ are orthogonal. On one hand, an argument similar to the proof of Claim 1 in Theorem 3.3 proves that if $\vec{x} \in \mathcal{N}(T)$, then $\vec{x}\downharpoonleft^{T}_{\mathfrak{A}} \in \mathcal{N}(\mathfrak{A})$ for $\mathfrak{A} \in \mathcal{F}_{\text{atom}}(T)$. On the other hand, for each $\mathfrak{A} \in \mathcal{F}_{\text{atom}}(T)$, let $\vec{x}(\mathfrak{A}) \in \mathcal{N}(\mathfrak{A})$. Since

$$A(T)\left(\sum_{\mathfrak{A}\in\mathcal{F}_{\text{atom}}(T)} \vec{x}(\mathfrak{A})\upharpoonleft^{T}_{\mathfrak{A}}\right) = \sum_{\mathfrak{A}\in\mathcal{F}_{\text{atom}}(T)} \left(A(\mathfrak{A})\vec{x}(\mathfrak{A})\right)\upharpoonleft^{T}_{\mathfrak{A}} = \theta,$$

we have $\bigoplus_{\mathfrak{A}\in\mathcal{F}_{\text{atom}}(T)} \subset \mathcal{N}(T)$.

Now we prove (4). Clearly the Similar arguments allow us to prove the following theorem. $\mathcal{R}(\mathfrak{A})\upharpoonleft^{T}_{\mathfrak{A}}$ are all orthogonal. Let u be a vertex of T. Let $\mathfrak{A} \in \mathcal{F}_{\text{atom}}(T)$ such that $u \in V(\mathfrak{A})$. Let $A(T)_{*\,u}$ be the column of $A(T)$, and let $A(\mathfrak{A})_{*\,u}$ be the column of $A(\mathfrak{A})$ both associated with the vertex u. Then

$$A(T)_{*\,u} = A(\mathfrak{A})_{*\,u}\upharpoonleft^{T}_{\mathfrak{A}} + \sum_{\{u,v\}\in BE(T)} e_v.$$

Hence, by Lemma 3.13, $\mathcal{R}(T) \subset \bigoplus_{\mathfrak{A}\in\mathcal{F}_{\text{atom}}(T)} \mathcal{R}(\mathfrak{A})\upharpoonleft^{T}_{\mathfrak{A}}$. The statement (3) follows from statement (2). ∎

Corollary 3.15. *Let T be an independent tree. Then*

$$\bigcup_{\mathfrak{A}\in\mathcal{F}_{atom}(T)} \mathcal{B}_{rk}(\mathfrak{A})\upharpoonleft^{S}_{\mathfrak{A}},$$

is a basis of $\mathcal{R}(T)$.

By $B_{\mathcal{C}}(\mathbb{R}^G)$ we denote the standard basis of $\mathbb{R}^G$. Let T be a tree. The **atom forest** of T, denoted by $\mathcal{F}_{\text{atom}}(T)$ is the forest

$$\bigcup_{S\in\mathcal{F}_{\text{indep}}(T)} \mathcal{F}_{\text{atom}}(S).$$

Corollary 3.16. *Let T be a tree. Then*

$$\bigcup_{N\in\mathcal{F}_{match}(T)} B_{\mathcal{C}}(\mathbb{R}^N)\upharpoonleft^{T}_{N} \cup \bigcup_{S\in\mathcal{F}_{indep}(T)} \left(\bigcup_{\mathfrak{A}\in\mathcal{F}_{atom}(T)} B_{\mathcal{R}}(\mathfrak{A})\upharpoonleft^{S}_{\mathfrak{A}}\right)\upharpoonleft^{T}_{S}$$

is a basis of $\mathcal{R}(T)$.

Similar arguments allow us to prove the following theorem.

Theorem 3.17. *If T is a tree, then*

$$\begin{aligned}
\mathcal{R}(T) &= \bigoplus_{N\in\mathcal{F}_{match}(T)} \mathcal{R}(N)\upharpoonleft^{T}_{N} \oplus \bigoplus_{\mathfrak{A}\in\mathcal{F}_{atom}(T)} \mathcal{R}(\mathfrak{A})\upharpoonleft^{T}_{\mathfrak{A}},\\
\mathrm{rk}(T) &= \sum_{N\in\mathcal{F}_{match}(T)} \mathrm{rk}(N) + \sum_{\mathfrak{A}\in\mathcal{F}_{atom}(T)} \mathrm{rk}(\mathfrak{A}),\\
\mathcal{N}(T) &= \bigoplus_{\mathfrak{A}\in\mathcal{F}_{atom}(T)} \mathcal{N}(\mathfrak{A})\upharpoonleft^{T}_{\mathfrak{A}},\\
\mathrm{null}(T) &= \sum_{\mathfrak{A}\in\mathcal{F}_{atom}(T)} \mathrm{null}(\mathfrak{A}).
\end{aligned}$$

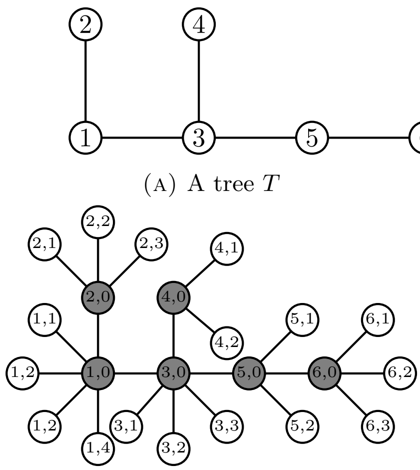
4. Graph operations closed in independent trees

Now we will define two graph operations under which the independent trees are closed. These operations are important because they allow thinking about independent trees without finding null spaces or a forest of atoms.

4.1. **Stellare.**

Definition 4.1. *Let G be a labeled graph of order n, with labels $[n]$. The $*(k_1,\dots,k_n)$-**stellare** of G is the graph obtained by adding $k_i \geqslant 2$ pendant vertices to vertex i of G.*

In the following, let $*G$ denote an arbitrary, but otherwise fixed, stellare of G. An example is shown in Figure 2. Instead of to say a stellare of a tree we just say a stellare tree.



(A) A tree T

(B) $*(4,3,3,2,2,3)T$

Figure 2. $*(4,3,3,2,2,3)T$ is a stellare tree of T

Theorem 4.2. *If T is a tree, then $*T$ is an independent tree, with* $\mathrm{Core}(*T) = V(T)$ *and* $\mathrm{Supp}(*T) = V(*T) - V(T)$.

Proof. Take $v \in V(*T) - V(T)$. By the stellare definition there exists $u \in V(T)$ and $w \in V(*T)$ such that $u \sim v$, the vertices v, w are neighbors of u in $*T$, and $w \notin V(T)$. Let $\vec{x}$ be a vector of $\mathbb{R}^{*T}$ such that

$$\vec{x}_i = \begin{cases} 1 & \text{if } i = v, \\ -1 & \text{if } i = w, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $A(*T)\,\vec{x} = \theta$. Hence, $V(*T)\backslash V(T) = \mathrm{Supp}(*T)$. By the stellare definition we have that $N[V(*T) - V(T)] = V(*T)$. Hence, we conclude that $*T$ is an independent tree, $\mathrm{Core}(*T) = V(T)$, and $\mathrm{Supp}(*T) = V(*T)\backslash V(T)$. ∎

Corollary 4.3. *Given a tree T with labels $[n]$, and $k_1, \ldots, k_n$ a list of n integers, each greater than or equal to 2. Then*

1. $\mathrm{null}(*(k_1, \ldots, k_n)T) = \sum_{i=1}^{n} k_i - n \geqslant n \geqslant \mathrm{null}(T)$, *where equality holds if and only if $n = 1$ and $k_1 = 2$,*
2. $\mathrm{rk}(*T) = 2n > \mathrm{rk}(T)$,
3. $\alpha(*(k_1, \ldots, k_n)T) = \sum_{i=1}^{n} k_i \geqslant 2n > \alpha(T)$,
4. $\nu(*T) = n > \nu(T)$,
5. $m(*(k_1, \ldots, k_n)T) = \prod_{i=1}^{n} k_i$,
6. $\gamma(*T) = n > \gamma(T)$, *and $V(T)$ is the only minimum (and total, if $n \geqslant 2$) dominating set of any $*T$.*

Proof. The statements 1 and 2 follow from Theorem 2.10 and Theorem 4.2. Statement 3 follows from Theorem 2.10, Theorem 4.2, and Theorem 2.10. Statements 4 and 5 follow from Theorem 2.10, Theorem 4.2, and Theorem 2.10. Statement 6 is clear. ∎

Theorem 4.4. *Let T be a tree and $*T$ a stellare of T. Then the set of vectors of $\mathbb{R}^{*T}$,*

$$\mathcal{B}_{rk}(T) := \{e_v, \vec{R}(v) \in \mathbb{R}^{(*T)} : v \in V(T)\},$$

is a basis of $\mathcal{R}(*T)$.

Proof. By Corollary 4.3, $\mathrm{rk}(*T) = 2v(T)$. Therefore, we only need to prove that the columns of the adjacency matrix of $*T$ are linear combinations of the vectors of $\mathcal{B}_{rk}$. For $v \in V(*T)$, let A_v denote the column of $A(*T)$ associated with the vertex v. Thus, if $v \in \mathrm{Supp}(*T)$, then $A_v = e_w$, for some $w \in \mathrm{Core}(*T) = V(T)$ and $w \sim v$. If $v \in \mathrm{Core}(*T) = V(T)$, then

$$A_v = e_{R(v)} + \sum_{\substack{w \in V(T) \\ w \sim v}} e_w.$$ ∎

Given a tree T with labels $[n]$, the **stellare labeling** of $*(k_1, \ldots, k_n)T$ is the set

$$\{(u, w) \ : \ u \in [n], \text{ and } w \in \{0, 1, \ldots, k_u\}\},$$

where the vertices labeled with $(u,0)$ are the core vertices of $*T$, and the vertices labeled with (u,w), where $w \in \{1,\dots,k_u\}$, are the supported vertices of $*T$ which are neighbors of u. See Figure 2.

Lemma 4.5. *Let T be a tree of order n and $*(k_1,\dots,k_n)T$ a stellare of T. Then the following set of vectors is a basis of the null space of $*(k_1,\dots,k_n)T$*

$$\mathcal{B}_{null}(T) := \{\vec{b}(i,j) \in \mathbb{R}^{*T} : i \in [n], j \in \{2,\dots,k_i\}\},$$

where

$$\vec{b}(i,j)_{(u,w)} = \begin{cases} 1 & \text{if } u = i \text{ and } w = 1, \\ -1 & \text{if } u = i \text{ and } w = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The set $\mathcal{B}_{null}(T)$ is a set of linear independent vectors. A direct computation shows that $A(*T)\vec{b} = \theta$ for all $\vec{b} \in \mathcal{B}_{null}(T)$. As $|\mathcal{B}_{null}(T)| = \mathrm{supp}(*T) - \mathrm{core}(*T)$, then by Theorem 2.10 the set $\mathcal{B}_{null}$ is a basis of the null space of $*(k_1,\dots,k_n)T$. ■

4.2. S-coalescence.

Definition 4.6. *Let $T_1,\dots,T_k$ be k disjoint independent trees. Let $v_i \in \mathrm{Supp}(T_i)$, for each $i \in [k]$. The* ***S-coalescence*** *of $(T_1,v_1),\dots,(T_k,v_k)$, denoted by*

$$\circledast_{i=1}^{k}(T_i,v_i)$$

is the tree obtained by identifying all the vertices v_i, denoting this single vertex by v^*. Let $N_{T_i}(v_i)$ be the neighborhood of v_i in T_i. Then $\circledast_{i=1}^{k}(T_i,v_i)$ is the tree with the set of vertices

$$V\left(\circledast_{i=1}^{k}(T_i,v_i)\right) = \left(\bigcup_{1\leqslant i\leqslant k}(V(T_i)-\{v_i\})\right) \cup \{v^*\},$$

and the set of edges

$$E\left(\circledast_{i=1}^{k}(T_i,v_i)\right) = \{\{u,v^*\} : u \in N_{T_i}(v_i)\} \cup \bigcup_{i=1}^{k} E(T_i) - \{\{u,v_i\} : u \in N_{T_i}(v_i)\}.$$

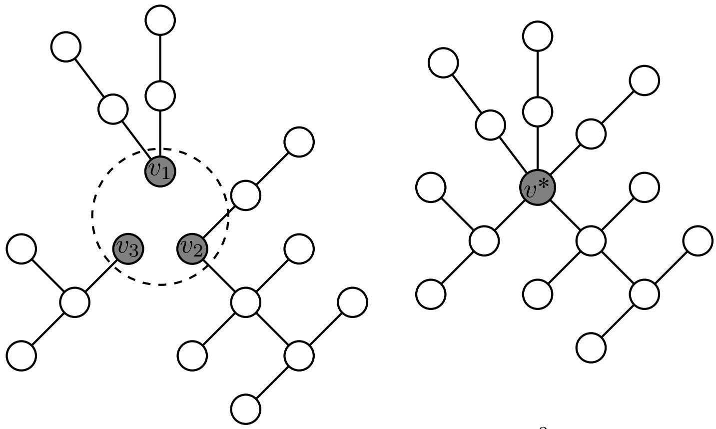
The following theorem proves that independent trees are closed under S-coalescence.

Theorem 4.7. *Let $T_1,\dots,T_k$ be k disjoint independent trees and $v_i \in \mathrm{Supp}(T_i)$. Then $\circledast_{i=1}^{k}(T_i,v_i)$ is an independent tree.*

Proof. It is left to the reader. ■

Corollary 4.8. *Let $T_1,\dots,T_k$ be k disjoint independent trees, and $v_i \in \mathrm{Supp}(T_i)$ for $i \in [k]$. Then*

(1) $\mathrm{Core}(\circledast_{i=1}^{k}(T_i,v_i)) = \bigcup_{i=1}^{k}\mathrm{Core}(T_i)$.
(2) $\mathrm{core}(\circledast_{i=1}^{k}(T_i,v_i)) = \sum_{i=1}^{k}\mathrm{core}(T_i)$.
(3) $\mathrm{Supp}(\circledast_{i=1}^{k}(T_i,v_i)) = \{v^*\} \cup \bigcup_{i=1}^{k}(\mathrm{Supp}(T_i)-\{v_i\})$.



(A) Independent trees T_1, T_2, and T_3

(B) $\underset{i=1}{\overset{3}{\circledast}}(T_i, v_i)$

FIGURE 3. S-coalescence of tree independent trees.

(4) $\operatorname{supp}(\circledast_{i=1}^k (T_i, v_i)) = 1 - k + \sum_{i=1}^k \operatorname{supp}(T_i)$.
(5) $\operatorname{rk}(\circledast_{i=1}^k (T_i, v_i)) = \sum_{i=1}^k \operatorname{rk}(T_i)$.
(6) $\operatorname{null}(\circledast_{i=1}^k (T_i, v_i)) = 1 - k + \sum_{i=1}^k \operatorname{null}(T_i)$.
(7) $\nu(\circledast_{i=1}^k (T_i, v_i)) = \sum_{i=1}^k \nu(T_i)$.
(8) $m(\circledast_{i=1}^k (T_i, v_i)) < \prod_{i=1}^k m(T_i)$.
(9) $\alpha(\circledast_{i=1}^k (T_i, v_i)) = 1 - k + \sum_{i=1}^k \alpha(T_i)$.

Proof. Clearly (1) and (4) follow from (3), and (2) follows from (1). Further, (5) follows from (2) and Theorem 2.10. The statement (6) follows from (2), (4), and (5). The statement (7) follows from Theorem 3.6 and (2). Finally, (9) follows from Theorem 2.10 and (4).

In order to prove (3), let $\vec{y} \in \operatorname{Supp}(\circledast_{i=1}^k (T_i, v_i))$. Without loss of generality we assume that $\vec{y}_{v^*} = 1$. As

$$A(T_i)\left((y \downarrow^{\circledast_{i=1}^k (T_i, v_i)}_{T_i - v_i}) \uparrow^{T_i}_{T_i - v_i} + e_{v_i}\right) = \theta,$$

where e_{v_i} is a canonical vector of $\mathbb{R}^{T_i}$. Thus,

$$\operatorname{Supp}(\underset{i=1}{\overset{k}{\circledast}}(T_i, v_i)) \subset \{v^*\} \cup \left(\bigcup_{i=1}^k \operatorname{Supp}(T_i) - \{v_i\}\right).$$

Hence, by the proof of Theorem 4.7, (3) follows.

To prove (8), just note that there is an injection between the maximum matchings of $\circledast_{i=1}^k (T_i, v_i)$ and $\prod_{i=1}^k M(T_i)$. Let $M \in \mathcal{M}(\circledast_{i=1}^k (T_i, v_i))$, and $u_i \in V(T_i)$

such that $\{u_i, v^*\} \in M$. Then

$$M - \{u_i, v^*\} + \{u_i, v_i\} \in \prod_{i=1}^{k} M(T_i).$$

But this injection is not onto. Let $M(i) \in \mathcal{M}(T_i)$ such that $v_i \in V(M(i))$. Clearly, every matching $M \in \mathcal{M}(\circledast_{i=1}^{k}(T_i, v_i))$ has cardinality lower than $\prod_{i=1}^{k} M(i)$. ∎

It is clear that any S-coalescence of an atom tree is an atom tree.

Proposition 4.9. *Atom trees are closed under S-coalescence.*

The set of all supported vertices with degree greater than one carries structural information about trees. They mark in the tree where the S-coalescences were made.

Theorem 4.10. *Let T be an independent tree. If $v \in \operatorname{Supp}(T)$ and $\deg(v) > 1$, then T is an S-coalescence of independent trees.*

Proof. It is left to the reader. ∎

We can apply the former decomposition a finite number of times in order to get the set of stellare trees that form the given independent tree T.

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