Sharp Bounds for the Annihilation Number of the Nordhaus-Gaddum type

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Let G be a graph with n vertices and G^c be its complement. The annihilation number of G, denoted by $a(G)$, is a graph invariant used as a sharp upper bound for the independence number. In this paper, we present the following bounds and Nordhaus-Gaddum type inequalities for the annihilation number

$$\left\lfloor \frac{n}{2} \right\rfloor \leq a(G) \leq n \qquad 2\left\lfloor \frac{n}{2} \right\rfloor \leq a(G) + a(G^c) \leq n + \left\lfloor \frac{n}{2} \right\rfloor.$$

We also investigate the extremal behavior of the invariant and showed that both parameters satisfy the interval property. In addition, we characterize some extremal graphs, ensuring that the bounds obtained are the best possible.

Keywords: Annihilation number, Nordhaus-Gaddum problem, Interval property, Extremal problems
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1 Introduction

The independence number of a graph is the cardinality of a largest set of mutually non-adjacent vertices. We observed that it is not always possible to determine the number of independence of a graph, since this is a well-known widely-studied NP-hard problem [10], and for this reason bounds for the independence number of a graph are investigated [11, 22]. Therefore, approximating this invariant through inequalities represents a relevant research topic.

The annihilation number of a graph G, denoted by $a(G)$, is a polynomial time computable upper bound for the independence number. It was originally defined by R. Pepper [21] in terms of a reduction process of the degree sequence that is associated with the process developed by Havel [14] and Hakimi [13] to determine when a given sequence of non-negative integers could be realized as the degree sequence of a graph, see in [12].

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While reading the original definition, Fajtlowicz noted that the largest integer k such that the sum of the smallest k degrees of graph G was at most its number of edges $e(G)$ is equivalent to the annihilation number $a(G)$ [18, 21], that is

$$a(G) = \max\left\{k \in \mathbb{N} : \sum_{i=1}^{k} d_i \leq e(G)\right\}.$$

The annihilation number and the independence number are used to investigate the relationship between the reactivity of an organic molecule, represented by a graph, and its independence number. More precisely, the research states that, for a fixed number of vertices, molecules with a lower number of independence are, in general, less reactive than molecules with a greater number of independence. This study is known in organic chemistry as the independence-stability hypothesis and was originally developed by S. Fajtlowicz [8, 9] and studied by R. Pepper [21].

In 1956, E. Nordhaus and J. Gaddum [20] gave lower and upper bounds on the sum and the product of the chromatic number of a graph and its complement in terms of the order of the graph. Since then, the Nordhaus-Gaddum problem is related to find lower and upper bounds to the following:

$$\rho(G) + \rho(G^c) \quad \text{and} \quad \rho(G)\rho(G^c),$$

where $\rho(G)$ is a graph invariant. In general these inequalities are quite elegant as they reveal extremal values for a graph parameter and its complement. On the other hand, it may be quite difficult to be obtained.

M. Aouchiche and P. Hansen [1] surveyed the Nordhaus-Gaddum type inequalities for various invariants of a graph, in particular, for invariants whose definitions are based on the cardinalities of particular subsets of the graph, such as the independence number, the domination number, the Roman domination number, the total domination number, among others. The relationship between these domination parameters and the annihilation number has been studied by several authors [3, 4, 5, 6, 7, 16, 19, 23], establishing a valuable connection with the Nordhaus-Gaddum inequalities for them.

In this paper, we present an upper and lower bound for the annihilation number of any graph G and prove that those bounds are the best possible. To state the results we denote by K_n and S_n the complete graph and the star graph on n vertices, respectively.

Theorem 1. *Let G a graph of order n. Then*

$$\left\lfloor \frac{n}{2} \right\rfloor \leq a(G) \leq n.$$

Equality hold in the upper bound if and only if G is isomorphic to nK_1.
If G is a non-empty k-regular graph then the equality hold in the lower bound.

Besides, as the main result, we show a solution to the Nordhaus-Gaddum problem for the annihilation number for any graph G and prove that those bounds are the best possible.

Theorem 2. *Let G a graph of order n. Then*

$$2\left\lfloor \frac{n}{2} \right\rfloor \leq a(G) + a(G^c) \leq n + \left\lfloor \frac{n}{2} \right\rfloor.$$

For n even, the equality hold in the upper bound if and only if G or G^c is isomorphic to nK_1.

For n odd, the equality hold in the upper bound if and only if G or G^c is isomorphic to nK_1 or $S_{d_n+1} \cup (n-d_n-1)K_1$, for $\left\lfloor \frac{n}{2} \right\rfloor \leq d_n \leq n-1$.
If G and G^c are non-empty graphs and G is a k-regular graph then the equality hold in the lower bound.

We emphasize that while the upper bounds have a simple expression, characterizing the extremal graphs is not straight, and that the lower bounds are satisfied for many graphs.

We say that a parameter of a graph satisfies the interval property if each integer value in an interval is realized by at least one graph. The interval property was studied recently in [2, 15, 17] and generalizes the behavior of a parameter in an interval making it a relevant research topic.

Using the results of Theorems 1 and 2 we establish the maximum and minimum values for both parameters investigated. As a consequence, we prove that the annihilation number and its Nordhaus-Gaddum problem associated with the sum satisfy the interval property.

The remainder of the paper is organized as follows. In the next section we present the notations and preliminaries used. The section 3 is devoted to prove the bounds for the annihilation number and its interval property. In section 4, we developed the necessary preliminaries to demonstrate the main result and, finally, we show a solution for the Nordhaus-Gaddum problem associated with the annihilation number and its interval property.

2 Notations and Preliminaries

Throughout this paper we will consider that $G = (V(G), E(G))$ is a simple graph of order $n = |V(G)|$ and size $e(G)$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges.

The complement of G, denoted by G^c, is the graph with $V(G^c) = V(G)$ and $E(G^c) = E(K_n) - E(G)$, in other words, it is the graph with the same vertex set such that two distinct vertices of G^c are adjacent if and only if they are not adjacent in G.

The degree d_i of a vertex v_i is the number of its neighbors, that is, it is the number of edges incident in v_i. The degree sequence of a graph G is given by $D(G) = (d_1 \leq \cdots \leq d_n)$, where d_i is the i-th smallest degree of G. A well-known result relates the sum of the degrees of a graph with its number of edges.

Lemma 3. *Let G a graph of order n then $\sum_{i=1}^{n} d_i = 2e(G)$.*

Since each edge defined by the n vertices of $V(G)$ is either in $E(G)$ or $E(G^c)$ it follows that the degree sequence of G^c is defined by $D(G)$ in the following form

$$\begin{aligned} D(G^c) &= (d_1^c \leq \cdots \leq d_i^c \leq \cdots \leq d_n^c) \\ &= (n-1-d_n \leq \cdots \leq n-1-d_{n+1-i} \leq \cdots \leq n-1-d_1). \end{aligned} \tag{1}$$

A k-regular graph is a graph in which every vertex has degree k. The relation between $D(G)$ and $D(G^c)$ provides the following result.

Lemma 4. *Let G a k-regular graph of order n then G^c is a $(n-k-1)$-regular graph.*

Our interest in interval property was briefly discussed in the introduction. In order to develop this investigation, we enunciate the definition of the interval property. Let $\mathcal{G}$ be a collection of graphs and $\xi : \mathcal{G} \to \mathbb{R}$ be a parameter of a graph defined on $\mathcal{G}$. We say that ξ has the interval property on $\mathcal{G}$ if $\xi(\mathcal{G}) = I \cap \mathbb{Z}$, for some interval $I \subset \mathbb{R}$.

3 Bounds for Annihilation Number

In this section we present bounds for the annihilation number $a(G)$ of a graph and prove that they are the best possible by characterizing the extremal graphs. Moreover, we show that the annihilation number satisfies the interval property.

Theorem 1. *Let G a graph of order n. Then*

$$\left\lfloor \frac{n}{2} \right\rfloor \leq a(G) \leq n.$$

Equality hold in the upper bound if and only if G is isomorphic to nK_1.
If G is a non-empty k-regular graph then the equality hold in the lower bound.

Proof. The upper bound follows trivially from the definition, so it remains to consider the equality case. Suppose $a(G) = n$, follows from the definition of annihilation number and by Lemma 3 that

$$2e(G) = \sum_{i=1}^{n} d_i \leq e(G),$$

this implies that $e(G) = 0$ and the only graph that satisfies this condition is $G = nK_1$.

To demonstrate the lower bound we proceed by contradiction. Suppose $a(G) = k < \left\lfloor \frac{n}{2} \right\rfloor$, using the definition of annihilation number and Lemma 3 we have

$$2e(G) = \sum_{i=1}^{n} d_i = \sum_{i=1}^{k+1} d_i + \sum_{i=k+2}^{n} d_i > e(G) + \sum_{i=k+2}^{n} d_i,$$

this implies that

$$\sum_{i=k+2}^{n} d_i < e(G) < \sum_{i=1}^{k+1} d_i,$$

a contradiction. So we conclude that $a(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$.

For the lower bound extremal cases, note that if G is a non-empty k-regular graph then Lemma 3 ensures that $e(G) = \frac{nk}{2}$, and this implies that $a(G) = \left\lfloor \frac{n}{2} \right\rfloor$. This example ensures that the lower bound obtained is the best possible. □

As a consequence, we can show that for each integer value in the interval $\left(\left\lfloor \frac{n}{2} \right\rfloor, n\right)$ there is at least one graph with this annihilation number, that is, the parameter satisfies the interval property.

Corollary 5. *Let n and k be integers such that $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq k \leq n-1$. If G is isomorphic to $(n-k)K_2 \cup (2k-n)K_1$, then $a(G) = k$.*

Proof. Suppose that $G = (n-k)K_2 \cup (2k-n)K_1$. Note that G has the following properties:
- $2n - 2k$ vertices of degree 1;
- $2k - n$ vertices of degree 0;
- $n - k$ edges.

Adding the k smallest degrees of G we have

$$\sum_{i=1}^{k} d_i = \sum_{i=1}^{2k-n} d_i + \sum_{i=2k-n+1}^{k} d_i = 0 + (n-k) = e(G).$$

This implies that the annihilation number of G is k and completes the proof. □

The lower bound presented in Theorem 1 was obtained by R. Pepper [21], however, the original proof uses the definition given by the reduction process of the degree sequence, while our demonstration uses the new approach given by the equivalent definition of Fajtlowicz, which is most appropriate to investigate the extremal behavior of the annihilation number.

4 Nordhaus-Gaddum for Annihilation Number

In this section we show a solution to the Nordhaus-Gaddum problem associated with the annihilation number and we characterize some of its extremal graphs. In the end, we demonstrate that the parameter $a(G) + a(G^c)$ satisfies the interval property.

First, we developed the preliminaries necessary to carry out the proof of the main result. For this, we characterize all graphs with annihilation number equal to $n-1$. This result has an important role in the characterization of extremal graphs of the Nordhaus-Gaddum problem.

Lemma 6. *Let G be a graph of order n with $a(G) = n-1$, then G is isomorphic to $S_{d_n+1} \cup (n-d_n-1)K_1$.*

Proof. From the definition of annihilation number and by Lemma 3 we have that

$$2e(G) = \sum_{i=1}^{n} d_i = \sum_{i=1}^{n-1} d_i + d_n \leq e(G) + d_n,$$

and this implies that $d_n \geq e(G)$.

By definition of degree we obtain that $d_n = e(G)$, consequently, we have that all the edges of G are incident on v_n. This condition characterizes the connected component induced by vertex v_n as the star graph S_{d_n+1} and we conclude that G is isomorphic to $S_{d_n+1} \cup (n-d_n-1)K_1$. □

In the following, we present a useful technical lemma to proof of the main result of this section.

Lemma 7. *Let G be a graph of order n, k an integer with $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ and denote*

$$x = k(n-1) - \sum_{i=n-k+1}^{n} d_i.$$

If $1 \leq x \leq k-1$ then $d_n = \cdots = d_{n-k+x+1} = n-1$ and $d_{n-k+x} \geq n-2$.

Proof. Note that $x = k(n-1) - \sum\limits_{i=n-k+1}^{n} d_i$ implies that

$$\sum_{i=n-k+x+1}^{n} (n-1-d_i) + \sum_{i=n-k+1}^{n-k+x} (n-2-d_i) = 0.$$

If $\sum\limits_{i=n-k+x+1}^{n} (n-1-d_i) > 0$, then $d_{n-k+x+1} \leq n-2$ and $\sum\limits_{i=n-k+1}^{n-k+x} (n-2-d_i) < 0$, a contradiction, because $d_{n-k+1} \leq \cdots \leq d_{n-k+x} \leq d_{n-k+x+1} \leq n-2$.

Therefore, $\sum_{i=n-k+x+1}^{n}(n-1-d_i)=0$ and this implies that $d_n=\cdots=d_{n-k+x+1}=n-1$.

Moreover, we have $\sum_{i=n-k+1}^{n-k+x}(n-2-d_i)=0$, which ensures $d_{n-k+x}\geq n-2$. $\square$

Finally, we present a result on the Nordhaus-Gaddum problem associated with the annihilation number.

Theorem 2. *Let G a graph of order n. Then*

$$2\left\lfloor\frac{n}{2}\right\rfloor\leq a(G)+a(G^c)\leq n+\left\lfloor\frac{n}{2}\right\rfloor.$$

For n even, the equality hold in the upper bound if and only if G or G^c is isomorphic to nK_1.
For n odd, the equality hold in the upper bound if and only if G or G^c is isomorphic to nK_1 or $S_{d_n+1}\cup(n-d_n-1)K_1$, for $\left\lfloor\frac{n}{2}\right\rfloor\leq d_n\leq n-1$.
If G and G^c are non-empty graphs and G is a k-regular graph then the equality hold in the lower bound.

Proof. The lower bound follows directly from the application of Theorem 1 in G and G^c. For the extremal cases, note that if G is a non-empty k-regular graph then G^c is a non-empty $(n-k-1)$-regular graph, from Theorem 1 we have that $a(G)=a(G^c)=\left\lfloor\frac{n}{2}\right\rfloor$. This example ensures that the lower bound is the best possible.

To demonstrate the upper bound we use the definition of annihilation number and the relation between the sequences of the degrees of G and G^c given in (1). So we have

$$a(G)(n-1)-\sum_{i=n+1-a(G)}^{n}d_i^c=\sum_{i=1}^{a(G)}d_i\leq e(G),\ \text{and}$$
$$a(G^c)(n-1)-\sum_{i=n+1-a(G^c)}^{n}d_i=\sum_{i=1}^{a(G^c)}d_i^c\leq e(G^c).$$

Adding the inequalities and using Lemma 3 we get

$$\begin{aligned}[a(G)+a(G^c)](n-1) &\leq e(G)+e(G^c)+\sum_{i=n+1-a(G^c)}^{n}d_i+\sum_{i=n+1-a(G)}^{n}d_i^c\\ &\leq e(G)+e(G^c)+\sum_{i=1}^{n}d_i+\sum_{i=1}^{n}d_i^c\\ &= e(G)+e(G^c)+2e(G)+2e(G^c)\\ &= \frac{n(n-1)}{2}+2\left\lceil\frac{n(n-1)}{2}\right\rceil.\end{aligned}$$

Note that $a(G)$ and $a(G^c)$ are integers, so simplifying the inequality above we get the result

$$a(G)+a(G^c)\leq\left\lfloor\frac{n}{2}\right\rfloor+n.$$

To characterize the extremal graphs remember that $a(G)=n$ if and only if G is isomorphic to nK_1 (Theorem 1) and, consequently, we have that $a(G^c)=a(K_n)=\left\lfloor\frac{n}{2}\right\rfloor$. Clearly this is an extremal case, moreover, it is the only case with $a(G)=n$.

If there is another graph G that satisfies the equality of the upper bound then it must have $a(G) = n - k$ and $a(G^c) = \left\lfloor \frac{n}{2} \right\rfloor + k$, for k integer with $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$ when n is even and $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$ when n is odd. We separated the characterization of the extremal graphs for the upper bound in cases.

Case 1: n is even.

We proceed by contradiction to show that no other graph satisfies the upper bound. Using the inequalities of the upper bound proof, we have

$$a(G)(n-1) - \sum_{i=n+1-a(G)}^{n} d_i^c = \sum_{i=1}^{a(G)} d_i \leq e(G), \text{ and } \sum_{i=1}^{a(G^c)} d_i^c \leq e(G^c).$$

Adding the inequalities and replacing $a(G)$ and $a(G^c)$ we get

$$(n-k)(n-1) \leq e(G) + e(G^c) + \sum_{i=k+1}^{n} d_i^c - \sum_{i=1}^{\frac{n}{2}+k} d_i^c.$$

Note that for $n \geq 2$ we have $\frac{n}{2} + k \geq k + 1$, so we obtain that

$$\begin{aligned} (n-k)(n-1) &\leq e(G) + e(G^c) + \sum_{i=\frac{n}{2}+k+1}^{n} d_i^c - \sum_{i=1}^{k} d_i^c \\ &= \frac{n(n-1)}{2} + \sum_{i=\frac{n}{2}+k+1}^{n} d_i^c - \sum_{i=1}^{k} d_i^c. \end{aligned}$$

Subcase 1.1: $\sum_{i=1}^{k} d_i^c = 0$.

In this case $d_1^c = \cdots = d_k^c = 0$, and this implies that $d_n^c \leq (n-1-k)$. So we have

$$\begin{aligned} (n-k)(n-1) &\leq \frac{n(n-1)}{2} + \sum_{i=\frac{n}{2}+k+1}^{n} d_i^c \\ &\leq \frac{n(n-1)}{2} + \left(\frac{n}{2} - k\right)(n-1-k), \end{aligned}$$

and we conclude that $k\left(\frac{n}{2} - k\right) \leq 0$, a contradiction due to $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.

Subcase 1.2: $\sum_{i=1}^{k} d_i^c \geq 1$.

Using that $d_k^c \leq n-1$ we have

$$\begin{aligned} (n-k)(n-1) &\leq \frac{n(n-1)}{2} + \sum_{i=\frac{n}{2}+k+1}^{n} d_i^c - \sum_{i=1}^{k} d_i^c \\ &\leq \frac{n(n-1)}{2} + \left(\frac{n}{2} - k\right)(n-1) - \sum_{i=1}^{k} d_i^c, \end{aligned}$$

and we conclude that $\sum_{i=1}^{k} d_i^c \leq 0$, a contradiction.

Case 2: n is odd.

Subcase 2.1: $k = 1$ **or** $k = \left\lfloor \frac{n}{2} \right\rfloor$**.**

In this subcase we find the remaining extremal graphs.

For $k = 1$ the Lemma 6 states that G is isomorphic to $S_{d_n+1} \cup (n - d_n - 1)K_1$, this ensures that G^c has one vertex with degree $n - d_n - 1$, d_n vertices with degree $n - 2$, $n - d_n - 1$ vertices with degree $n - 1$ and $\frac{n(n-1)}{2} - d_n$ edges.

Using that $a(G^c) = \left\lfloor \frac{n}{2} \right\rfloor + 1$, we have

$$(n-d_n-1)+d_n(n-2)+\left[\left\lfloor \frac{n}{2} \right\rfloor + 1 - (d_n+1)\right](n-1) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1} d_i^c \leq e(G^c) = \frac{n(n-1)}{2} - d_n,$$

this provides $\left\lfloor \frac{n}{2} \right\rfloor \leq d_n \leq n - 1$. Therefore, we conclude that G is isomorphic to $S_{d_n+1} \cup (n - d_n - 1)K_1$, for $\left\lfloor \frac{n}{2} \right\rfloor \leq d_n \leq n - 1$.

For $k = \left\lfloor \frac{n}{2} \right\rfloor$ we have $a(G^c) = n - 1$ and we proceed in an analogous way to the previous case, with G^c in the role of G.

Subcase 2.2: $2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$**.**

In this subcase we prove that there are no more extremal graphs.

Suppose without loss of generality that G is connected and $a(G) = n - k$.

Using the relation between the sequences of the degrees of G and G^c given in (1) we obtain

$$\begin{aligned}
\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor + k} d_i^c &= \left(\left\lfloor \frac{n}{2} \right\rfloor + k\right)(n-1) - \sum_{i=\lfloor \frac{n}{2} \rfloor - k + 2}^{n} d_i \\
&= \left(\left\lfloor \frac{n}{2} \right\rfloor + k\right)(n-1) - \sum_{i=n-k+1}^{n} d_i - \sum_{i=\lfloor \frac{n}{2} \rfloor - k + 2}^{n-k} d_i \\
&= \left(\left\lfloor \frac{n}{2} \right\rfloor + k\right)(n-1) - \sum_{i=n-k+1}^{n} d_i - \sum_{i=1}^{n-k} d_i + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor - k + 1} d_i.
\end{aligned}$$

Using that $a(G) = n - k$ and the definition of annihilation number

$$\begin{aligned}
\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor + k} d_i^c &\geq \left(\left\lfloor \frac{n}{2} \right\rfloor + k\right)(n-1) - e(G) - \sum_{i=n-k+1}^{n} d_i + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor - k + 1} d_i \\
&= \left[n\left\lfloor \frac{n}{2} \right\rfloor - e(G)\right] - \left\lfloor \frac{n}{2} \right\rfloor + k(n-1) - \sum_{i=n-k+1}^{n} d_i + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor - k + 1} d_i
\end{aligned}$$

Finally, simplifying the inequality above we obtain

$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor + k} d_i^c \geq e(G^c) + \left[k(n-1) - \sum_{i=n-k+1}^{n} d_i\right] - \left\lfloor \frac{n}{2} \right\rfloor + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor - k + 1} d_i. \tag{2}$$

In the subcases that follow we study the values of $k(n-1) - \sum_{i=n-k+1}^{n} d_i$.

Subcase 2.2.1: $k(n-1) - \sum_{i=n-k+1}^{n} d_i > k-1$**.**

Since G is connected we have

$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor - k + 1} d_i \geq \left\lfloor \frac{n}{2} \right\rfloor - k + 1.$$

Applying in the inequality (2) we get

$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor + k} d_i^c > e(G^c) + (k-1) - \left\lfloor \frac{n}{2} \right\rfloor + \left(\left\lfloor \frac{n}{2} \right\rfloor - k + 1 \right) = e(G^c).$$

This ensures that $a(G^c) < \lfloor \frac{n}{2} \rfloor + k$ and the result follows.

Subcase 2.2.2: $k(n-1) - \sum_{i=n-k+1}^{n} d_i = 0$**.**

In this case we have $d_n = \cdots = d_{n-k+1} = n-1$ and, consequently, $d_1 \geq k$. This implies that

$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor - k + 1} d_i \geq k \left(\left\lfloor \frac{n}{2} \right\rfloor - k + 1 \right).$$

Applying in the inequality (2) we get

$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor + k} d_i^c \geq e(G^c) + 0 - \left\lfloor \frac{n}{2} \right\rfloor + k \left(\left\lfloor \frac{n}{2} \right\rfloor - k + 1 \right) \geq e(G^c) + \left\lfloor \frac{n}{2} \right\rfloor - 2.$$

For $n \geq 7$ the previous inequality ensures that $a(G^c) < \lfloor \frac{n}{2} \rfloor + k$. The remaining cases ($n \leq 5$) are easily checked.

Subcase 2.2.3: $1 \leq k(n-1) - \sum_{i=n-k+1}^{n} d_i \leq k-1$**.**

To simplify the notation, we denote $x = k(n-1) - \sum_{i=n-k+1}^{n} d_i$.

Lemma 7 states that $d_n = \cdots = d_{n-k+x+1} = n-1$ and $d_{n-k+x} \geq n-2$, consequently, we have that $d_1 \geq k-x$ and $d_2 \geq k-x+1$. Applying this in the inequality (2) we get

$$\begin{aligned}
\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor + k} d_i^c &\geq e(G^c) + x - \left\lfloor \frac{n}{2} \right\rfloor + (k-x) + \left(\left\lfloor \frac{n}{2} \right\rfloor - k \right)(k-x+1) \\
&= e(G^c) + \left(\left\lfloor \frac{n}{2} \right\rfloor - k \right)(k-x) \\
&\geq e(G^c) + 1 > e(G^c).
\end{aligned}$$

This ensures that $a(G^c) < \lfloor \frac{n}{2} \rfloor + k$ and the result follows.

Thus, we conclude that there are no graphs with $a(G) = n-k$ and $a(G^c) = \lfloor \frac{n}{2} \rfloor + k$, for $2 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$, that satisfy the upper bound presented. This concludes the proof. □

Similar to the annihilation number, we can show that for each integer in the interval defined in Theorem 2 there is at least one graph G for which $a(G) + a(G^c)$ assumes this value.

Corollary 8. *Let n and k be integers such that*

$$2\left\lfloor\frac{n}{2}\right\rfloor + 1 \leq k \leq n + \left\lfloor\frac{n}{2}\right\rfloor - 1.$$

If G is isomorphic to $\left(n + \left\lfloor\frac{n}{2}\right\rfloor - k\right) K_2 \cup \left(2k - 2\left\lfloor\frac{n}{2}\right\rfloor - n\right) K_1$, then $a(G) + a(G^c) = k$.

Proof. Suppose that $G = \left(n + \left\lfloor\frac{n}{2}\right\rfloor - k\right) K_2 \cup \left(2k - 2\left\lfloor\frac{n}{2}\right\rfloor - n\right) K_1$. Note that G has the following properties:

- $2n + 2\left\lfloor\frac{n}{2}\right\rfloor - 2k$ vertices of degree 1;
- $2k - 2\left\lfloor\frac{n}{2}\right\rfloor - n$ vertices of degree 0;
- $n + \left\lfloor\frac{n}{2}\right\rfloor - k$ edges.

Adding the $k - \left\lfloor\frac{n}{2}\right\rfloor$ smallest degrees of G we have

$$\sum_{i=1}^{k-\left\lfloor\frac{n}{2}\right\rfloor} d_i = \sum_{i=1}^{2k-2\left\lfloor\frac{n}{2}\right\rfloor-n} d_i + \sum_{i=2k-2\left\lfloor\frac{n}{2}\right\rfloor-n+1}^{k-\left\lfloor\frac{n}{2}\right\rfloor} d_i = 0 + \left(n + \left\lfloor\frac{n}{2}\right\rfloor - k\right) = e(G).$$

This implies that the annihilation number of G is $k - \left\lfloor\frac{n}{2}\right\rfloor$.

Using the definition of complement graph and the relation between the sequences of the degrees of G and G^c given in (1), we have that G^c has the following properties:

- $2k - 2\left\lfloor\frac{n}{2}\right\rfloor - n$ vertices of degree $n - 1$;
- $2n + 2\left\lfloor\frac{n}{2}\right\rfloor - 2k$ vertices of degree $n - 2$;
- $\frac{n(n-1)}{2} - n - \left\lfloor\frac{n}{2}\right\rfloor + k$ edges.

Adding the $\left\lfloor\frac{n}{2}\right\rfloor$ smallest degrees of G^c we have

$$\sum_{i=1}^{\left\lfloor\frac{n}{2}\right\rfloor} d_i^c = \left\lfloor\frac{n}{2}\right\rfloor (n-2) \leq \frac{n(n-1)}{2} - n - \left\lfloor\frac{n}{2}\right\rfloor + k = e(G^c).$$

This implies that the annihilation number of G^c is $\left\lfloor\frac{n}{2}\right\rfloor$.

Consequently, we conclude that $a(G) + a(G^c) = k$. □

To conclude, we emphasize the expressions for the lower and upper bounds are simple and elegant, but it seems to be hard to characterize the graphs satisfying the extremal values. In particular, we observed that the lower bound is satisfied by a large number of graphs and to carry out its characterization is possible future work that will be useful in understanding the extreme behavior of the annihilation number.

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